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RESEARCH ARTICLE

Inductive inference from weakly consistent belief bases

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Abstract
We consider nonmonotonic inferences from belief bases that contain conditionals enforcing some of the possible worlds to be infeasible and thus completely implausible. In contrast to belief bases satisfying the strong notion of consistency requiring every world to be at least somewhat plausible, we call such belief bases weakly consistent. First, we review the treatment of weakly consistent belief bases by the seminal approaches of p-entailment, which coincides with system P, and of system Z, which coincides with rational closure. Then we focus on c-inference, an inductive inference operator that has been shown to exhibit many desirable properties put forward for nonmonotonic reasoning. It is based on c-representations, which are a special kind of ranking model ordering worlds according to their plausibility. While c-representation is defined for strongly consistent belief bases only, in this article, we extend the notions of c-representation and of c-inference to cover also weakly consistent belief bases. We adapt a constraint satisfaction problem (CSP) characterizing c-representations to capture extended c-representations, and we show how this extended CSP can be used to characterize extended c-inference, providing a basis for its implementation. We show various properties of extended c-inference and in particular, we prove that also the extended notion of c-inference fully satisfies syntax splitting. Furthermore, we extend and evaluate credulous and weakly skeptical c-inference to weakly consistent belief bases and provide characterizations for them as CSPs.

1. Introduction

Conditionals of the form ‘if A then usually B’ state a conneQction between the antecedent A and the consequent B that is plausible, but that also carries uncertainty and is defeasible. A semantics for conditional belief bases Δ consisting of a set of conditionals typically provides a means for ordering the underlying possible worlds Ω according to their plausibility. Ranking functions κ (Spohn, 1988) do this by assigning a degree of implausibility to the worlds via a mapping $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$; the higher the rank $\kappa(\omega)$ of a possible world ω, the less plausible or the more surprising it is, with a rank of infinity indicating that a world is completely implausible. A ranking function κ models a conditional ‘if A then usually B’, formally denoted by $(B|A)$, if the verification of $(B|A)$ is strictly more plausible than its falsification, that is, if $\kappa(AB) < \kappa(A\overline{B})$.

The well-known consistency test for conditional belief bases given by Goldszmidt and Pearl (1996) yields a rather strong notion of consistency. It requires that for Δ to be consistent there must exist a ranking function κ modelling all conditionals in Δ such that κ assigns to each world ω a natural number $\kappa(\omega) < \infty$. Thus, this notion of consistency, which we will refer to as *strong consistency* in the following, enforces that every world has at least some plausibility, that is, prohiting any completely implausible world ω with $\kappa(\omega) = \infty$. Here, we will take a broader view on consistency of belief bases that

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allows that only a subset of worlds is considered feasible, while other worlds may be infeasible and thus fully implausible, expressed by $\kappa(\omega) = \infty$. Belief bases satisfying this generalized notion of consistency are called *weakly consistent* (Haldimann *et al.*, 2023). Contrary to strongly consistent belief bases that only contain defeasible beliefs, weakly consistent belief bases can also contain *strict* beliefs (Casini & Straccia, 2013). Such strict beliefs correspond to the requirement that some worlds are infeasible, forcing all worlds ω not satisfying a strict belief to have rank $\kappa(\omega) = \infty$.

The significance of weakly consistent belief bases is given by their increased expressive power. In a strongly consistent belief base, one can state that a formula A holds plausibly, but one can not express that A definitely holds and that therefore $\neg A$ is completely ruled out, while this can be expressed straightforwardly in a weakly consistent belief base. Thus, using the concept of weak consistency, one can also realize reasoning from belief bases that contain both conditionals stating defeasible beliefs and formulas in classical logic stating undefeasible strict beliefs.

Inductive inference from a conditional belief base Δ means to complete the set of explicit beliefs given in Δ to an inference relation representing the beliefs entailed by Δ. The resulting inference relation will contain both the beliefs stated explicitly in Δ as well as the beliefs entailed by Δ. Because Δ contains defeasible beliefs, the resulting relation is a nonmonotonic inference relation. There is no generally accepted 'best' answer to the question what a belief base should entail (Lehmann & Magidor, 1992). For formally capturing the process of inductive inference from Δ, the notion of an *inductive inference operator* mapping each Δ to an inference relation $\mid\!\sim_{\Delta}$ has been introduced (Kern-Isberner *et al.*, 2020). Employing the viewpoint of inductive inference operators, we will consider various inference methods in this article, starting with the seminal p-entailment, that can be characterized by system P (Adams, 1975; Kraus *et al.*, 1990), and Pearl's system Z that has been shown to coincide with rational closure (Lehmann, 1989; Goldszmidt & Pearl, 1996). We recall how these inductive inference operators deal with weakly consistent belief bases and how characterizations of them can be used for implementing them.

We then focus on the *c-representations* (Kern-Isberner, 2001, 2004) of a belief base Δ, a special kind of ranking functions modelling Δ. c-Representations define inductive inference operators that satisfy most advanced properties of nonmonotonic inference, particularly syntax splitting and conditional syntax splitting (Kern-Isberner *et al.*, 2020; Heyninck *et al.*, 2023; Beierle *et al.*, 2024). While initially introduced only for belief bases satisfying the strong notion of consistency described above, in this paper, we define extended c-representations that also cover belief bases satisfying the weaker notion of consistency where some possible worlds may be assigned a rank of ∞ indicating them to be completely infeasible according to Δ. This allows for realizing a kind of paraconsistent conditional reasoning based on the strong structural concept of c-representations. The notion of c-inference was introduced in Beierle *et al.* (2016a 2018) as nonmonotonic inference taking all c-representations into account, inheriting the restriction that it is only defined for strongly consistent belief bases. Using extended c-representations, we will introduce an extended version of c-inference that also covers weakly consistent belief bases.

The c-representations of a belief base Δ can be characterized by a CSP, and in Beierle *et al.* (2016a); 2018), it is shown that c-inference can also be realized by a CSP. Here, we develop both a CSP that characterizes all extended c-representations and a simplified version of this CSP the solutions of which still cover all c-representations relevant for c-inference. Furthermore, we show how extended c-inference can be realized by a CSP.

An inductive inference operator satisfies syntax splitting if it satisfies the axioms of independence (Ind) and relevance (Rel) (Kern-Isberner *et al.*, 2020). While these axioms have been developed taking only strongly consistent belief bases into account, here we employ versions of (Ind) and (Rel) adapted to our setting of weakly consistent belief bases and prove that extended c-inference complies with syntax splitting.

While c-inference is skeptical in the sense that it takes every c-representation of Δ into account, also credulous and weakly skeptical c-inference have been introduced, the latter lying between the cautious skeptical and the bold credulous reasoning method (Beierle *et al.*, 2018; Beierle *et al.*, 2021). We introduce extended versions of these inductive inference methods that cover also weakly consistent belief bases, characterize them by CSPs that can be used for implementations, and show how they relate to the unextended versions requiring strong consistency.

In summary, the main contributions of this article are

- Compactly presenting the notion of weak consistency and how its treatment by p-entailment (or system P) and system Z can be characterized and implemented;
- Generalizing the concept of c-representations to cover also weakly consistent belief bases and characterizing them as solutions of a CSP;
- Extending c-inference by taking all extended c-representations of a belief base into account to cover also weakly consistent belief bases;
- Modelling extended c-inference by a CSP, thus providing a basis for implementations;
- Proving that extended c-inference fully complies with the syntax splitting postulates for inductive inference operators;
- Extending credulous and weakly skeptical c-inference to cover also weakly consistent belief bases, modelling them by CSPs, and evaluating the resulting inductive inference operators.

This article is a revised and largely extended version of our FoIKS-2024 conference paper Haldimann *et al*. (2024). The additions include in particular the following aspects. We added full proofs for all propositions and lemmas that have been left out in the conference paper. We elaborate methods for checking weak consistency of a belief base and compactly recall how p-entailment and system Z handle weakly consistent belief bases. We investigate conditional indifference of extended c-representations and evaluate extended c-inference with respect to semi-monotony and rational monotony and weakened versions thereof. Furthermore, we develop extended versions of credulous and weakly skeptical c-inference and their modelling as CSPs, and we provide a map of all arising inference operators and their interrelationships.

The remainder of this article is organized as follows: We recall the background on conditional logic in Section 2 and inductive inference in Section 3. In Section 4, we elaborate and illustrate the different kinds of consistency and address the treatment of weakly consistent belief bases by p-entailment and system Z. We develop extended c-representations in Section 5, extended c-inference in Section 6, and present a corresponding CSP-characterizations in Section 7. In Section 8, we address syntax splitting, and Section 9 deals with credulous and weakly skeptical inference. In Section 10, we conclude and point out future work.

2. Conditional logic

A *(propositional) signature* is a finite set Σ of propositional variables. Assuming an underlying signature Σ, we denote the resulting propositional language by $\mathcal{L}_\Sigma$. Usually, we denote elements of signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. The set of interpretations over the underlying signature is denoted as Ω_Σ. Interpretations are also called *worlds* and Ω_Σ the *universe*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}$ if A holds in ω, denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $Mod_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$ or short as Ω_A. The Σ in Ω_Σ, $\mathcal{L}_\Sigma$ and $Mod_\Sigma(A)$ can be omitted if the signature is clear from the context or if the underlying signature is not relevant. A formula A *entails* a formula B, denoted by $A \models B$, if $\Omega_A \subseteq \Omega_B$. A formula A is *satisfiable* if $\Omega_A \neq \emptyset$. By slight abuse of notation we sometimes interpret worlds as the corresponding complete conjunction of all elements in the signature in either positive or negated form.

A *conditional* $(B|A)$ connects two formulas A, B and represents the rule 'If A then usually B', where A is called the *antecedent* and B the *consequent* of the conditional. The conditional language is denoted as $(\mathcal{L}|\mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}_\Sigma\}$. A finite set of conditionals is called a *belief base*. We use a three-valued semantics of conditionals in this paper (de Finetti, 1937). For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Popular models for belief bases are ranking functions (also called ordinal conditional functions, OCF) (Spohn, 1988, 2012) and total preorders (TPO) on Ω_Σ (Darwiche & Pearl, 1997). An OCF $\kappa : \Omega_\Sigma \to \mathbb{N} \cup \{\infty\}$ maps worlds to a *rank* such that at least one world has rank 0, that is, $\kappa^{-1}(0) \neq \emptyset$. OCFs have been first introduced by Spohn (1988) in a more general form. The intuition is that worlds with lower ranks are more plausible than worlds with higher ranks; worlds with rank ∞ are considered infeasible. OCFs are lifted to formulas by mapping a formula A to the smallest rank of a model of A, or to ∞ if A has no models. An OCF κ is a model of a conditional $(B|A)$, denoted as $\kappa \models (B|A)$, if $\kappa(A) = \infty$ or if $\kappa(AB) < \kappa(A\overline{B})$; κ is a model of a belief base Δ, denoted as $\kappa \models \Delta$, if it is a model of every conditional in Δ. For $\Sigma' \subseteq \Sigma$ the *marginalization* of a ranking function $\kappa : \Omega_\Sigma \to \mathbb{N} \cup \{\infty\}$ to Σ' is the ranking function $\kappa_{|\Sigma'} : \Omega_{\Sigma'} \to \mathbb{N} \cup \{\infty\}$ defined by $\kappa_{|\Sigma'}(\omega') = \min\{\kappa(\omega) \mid \omega_{|\Sigma'} = \omega'\}$.

Lemma 1. *Let $\kappa : \Omega_\Sigma \to \mathbb{N} \cup \{\infty\}$ be a ranking function. Let $\Sigma' \subseteq \Sigma$ and let $\kappa' = \kappa_{|\Sigma'}$. Then, for any formula $A \in \mathcal{L}_{\Sigma'}$ we have that $\kappa(A) = \kappa'(A)$.*

Proof. We can obtain the models of A with respect to Σ by extending the models of A with respect to Σ' by any possible valuation of $\Sigma \setminus \Sigma'$, that is, $Mod_\Sigma(A) = \{(\omega^a \cdot \omega^b) \mid \omega^a \in Mod_{\Sigma'}(A), \omega^b \in \Omega_{\Sigma\setminus\Sigma'}\}$. By definition, we have that $\kappa'(\omega') = \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma, \omega_{|\Sigma'} = \omega'\} = \min\{\kappa(\omega' \cdot \omega^b) \mid \omega^b \in \Omega_{\Sigma\setminus\Sigma'}\}$. Therefore,

$$\begin{aligned} \kappa(A) &= \min\{\kappa(\omega^a \cdot \omega^b) \mid \omega^a \in Mod_{\Sigma'}(A), \omega^b \in \Omega_{\Sigma\setminus\Sigma'}\} \\ &= \min\{\kappa'(\omega') \mid \omega' \in Mod_{\Sigma'}(A)\} = \kappa'(A). \end{aligned}$$

□

Lemma 2. *Let $\kappa : \Omega_\Sigma \to \mathbb{N} \cup \{\infty\}$ be a ranking function. Let $\Sigma' \subseteq \Sigma$ and let $\kappa' = \kappa_{|\Sigma'}$. Then, for formulas $A, B \in \mathcal{L}_{\Sigma'}$ we have that $\kappa \models (B|A)$ iff $\kappa' \models (B|A)$.*

Proof. Using Lemma 1, we have $\kappa(A) = \kappa'(A)$ and $\kappa(AB) = \kappa'(AB)$ and $\kappa(A\overline{B}) = \kappa'(A\overline{B})$. Therefore, $\kappa \models (B|A)$ iff $\kappa' \models (B|A)$. □

It is also possible to combine ranking functions κ_1, κ_2 over disjoint subsignatures Σ_1, Σ_2. For $\kappa_1 : \Omega_{\Sigma_1} \to \mathbb{N} \cup \{\infty\}$ and $\kappa_2 : \Omega_{\Sigma_2} \to \mathbb{N} \cup \{\infty\}$, the combination of κ_1 and κ_2, denoted by $\kappa_\oplus = \kappa_1 \oplus \kappa_2$, is defined by $\kappa_\oplus(\omega) = \kappa_1(\omega_{|\Sigma_1}) + \kappa_2(\omega_{|\Sigma_2})$.

Lemma 3. *Let Σ_1, Σ_2 be disjoint, let $\kappa_1 : \Omega_{\Sigma_1} \to \mathbb{N} \cup \{\infty\}$ and $\kappa_2 : \Omega_{\Sigma_2} \to \mathbb{N} \cup \{\infty\}$, and let $\kappa_\oplus = \kappa_1 \oplus \kappa_2$. For formulas $A \in \mathcal{L}_{\Sigma_1}, B \in \mathcal{L}_{\Sigma_2}$ it holds that $\kappa_\oplus(AB) = \kappa_1(A) + \kappa_2(B)$.*

Proof. Let $\Sigma = \Sigma_1 \cup \Sigma_2$. Because $A \in \mathcal{L}_{\Sigma_1}$ and $B \in \mathcal{L}_{\Sigma_2}$, we have that $Mod_\Sigma(AB) = \{(\omega^1 \cdot \omega^2) \mid \omega^1 \in Mod_{\Sigma_1}(A), \omega^2 \in Mod_{\Sigma_2}(B)\}$. Therefore, it holds that

$$\begin{aligned} \kappa_\oplus(AB) &= \min\{\kappa_\oplus(\omega) \mid \omega \in Mod_\Sigma(AB)\} \\ &= \min\{\kappa_1(\omega^1) + \kappa_2(\omega^2) \mid \omega^1 \in Mod_{\Sigma_1}(A), \omega^2 \in Mod_{\Sigma_2}(B)\} \\ &= \min\{\kappa_1(\omega^1) \mid \omega^1 \in Mod_{\Sigma_1}(A)\} + \min\{\kappa_2(\omega^2) \mid \omega^2 \in Mod_{\Sigma_2}(B)\} \\ &= \kappa_1(A) + \kappa_2(B). \end{aligned}$$

□

Lemma 4 (Kern-Isberner & Brewka, 2017). *Let Σ_1, Σ_2 be disjoint signatures and let $\kappa_1 : \Omega_{\Sigma_1} \to \mathbb{N} \cup \{\infty\}$ and $\kappa_2 : \Omega_{\Sigma_2} \to \mathbb{N} \cup \{\infty\}$ be ranking functions. Let $\kappa_\oplus = \kappa_1 \oplus \kappa_2$. Then $\kappa_{\oplus|\Sigma_1} = \kappa_1$.*

Proof. Let ω^1 be any world in Ω_{Σ_1}. Let $\omega^2 \in \Omega_{\Sigma_1}$ such that $\kappa_2(\omega^2) = 0$. We have that

$$\begin{aligned} \kappa_{\oplus|\Sigma_1}(\omega^1) &= \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma, \omega_{|\Sigma_1} = \omega^1\} \\ &= \min\{\kappa_1(\omega^1) + \kappa_2(\omega_{|\Sigma_2}) \mid \omega \in \Omega_\Sigma\} \\ &\overset{*}{=} \kappa_1(\omega^1). \end{aligned}$$

Equation $(*)$ holds because we can choose, for example, $\omega = (\omega^1 \cdot \omega^2)$, resulting in $\kappa_2(\omega_{|\Sigma_2}) = 0$. □

Lemmas 1–4 will be helpful when proving properties regarding syntax splitting in the sequel.

(REF)	*Reflexivity*	for all $A \in \mathcal{L}$ it holds that $A \mid\!\sim A$
(LLE)	*Left Logical Equivalence*	$A \equiv B$ and $B \mid\!\sim C$ imply $A \mid\!\sim C$
(RW)	*Right weakening*	$B \models C$ and $A \mid\!\sim B$ imply $A \mid\!\sim C$
(CM)	*Cautious Monotony*	$A \mid\!\sim B$ and $A \mid\!\sim C$ imply $AB \mid\!\sim C$
(CUT)		$A \mid\!\sim B$ and $AB \mid\!\sim C$ imply $A \mid\!\sim C$
(OR)		$A \mid\!\sim C$ and $B \mid\!\sim C$ imply $(A \vee B) \mid\!\sim C$

Figure 1. *The system P postulates for nonmonotonic inference relations.*

3. Inductive inference

The conditional beliefs of an agent are formally captured by a binary relation $\mid\!\sim$ on propositional formulas with $A \mid\!\sim B$ representing that A (defeasibly) entails B; this relation is called *inference* or *entailment relation*. Different sets of properties for inference relations have been suggested, and often the set of postulates called *system P* (Adams, 1975; Kraus *et al*., 1990) is considered as minimal requirement for inference relations. Inference relations satisfying system P are called *preferential inference relations*, for details we refer to Adams (1975), Kraus *et al*. (1990), see Figure 1.

Every ranking function κ induces a preferential inference relation $\mid\!\sim_\kappa$ by

$$A \mid\!\sim_\kappa B \quad \text{iff} \quad \kappa(A) = \infty \quad \text{or} \quad \kappa(AB) < \kappa(A\overline{B}). \tag{1}$$

Note that the condition $\kappa(A) = \infty$ in (1) ensures that system P's axiom (Reflexivity): $A \mid\!\sim_\kappa A$ is satisfied for $A \equiv \bot$.

Regarding the inference induced by a marginalized ranking function, the following lemma holds.

Lemma 5. *Let $\kappa : \Omega_\Sigma \to \mathbb{N} \cup \{\infty\}$ be a ranking function. Let $\Sigma' \subseteq \Sigma$ and let $\kappa' = \kappa_{|\Sigma'}$. Then, for formulas $A, B \in \mathcal{L}_{\Sigma'}$ we have that $A \mid\!\sim_\kappa B$ iff $A \mid\!\sim_{\kappa'} B$.*

Proof. Using Lemma 1 we have $\kappa(A) = \kappa'(A)$ and $\kappa(AB) = \kappa'(AB)$ and $\kappa(A\overline{B}) = \kappa'(A\overline{B})$. Therefore, $A \mid\!\sim_\kappa B$ iff $A \mid\!\sim_{\kappa'} B$. □

Inductive inference is the process of completing a given belief base to an inference relation. To formally capture this, we use inductive inference operators.

Definition 6 (inductive inference operator, Kern-Isberner et al., 2020). *An inductive inference operator is a mapping $C : \Delta \mapsto \mid\!\sim_\Delta$ that maps each belief base to an inference relation s.t. direct inference (DI) and trivial vacuity (TV) are fulfilled, that is,*

(DI) *if $(B|A) \in \Delta$ then $A \mid\!\sim_\Delta B$, and*
(TV) *if $\Delta = \emptyset$ and $A \mid\!\sim_\Delta B$, then $A \models B$.*

An inductive inference operator C is a *preferential inductive inference operator* if every inference relation $\mid\!\sim_\Delta$ in the image of C satisfies system P.

p-Entailment (Adams, 1975; Kraus *et al*., 1990) $C^p : \Delta \mapsto \mid\!\sim^p_\Delta$ is the most cautious preferential inductive inference operator. It is characterized by system P in the way that it only licenses inferences that can be obtained by iteratively applying the rules of system P to the belief base. Every other preferential inductive inference operator extends p-entailment. While extending p-entailment and adding some more inferences to the induced inference relations is usually desired, p-entailment can act as a basic guidance for inferences of the form $A \mid\!\sim \bot$ which can be seen as representations of 'strict' beliefs (i.e., A is completely unfeasible).

Postulate (Classic Preservation) 1. (adapted from Casini et al., 2019). *An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies (Classic Preservation) if for all belief bases Δ and $A \in \mathcal{L}$ it holds that $A \mid\!\sim_\Delta \bot$ iff $A \mid\!\sim^p_\Delta \bot$.*

4. Consistency of belief bases

There are different definitions of consistency of a belief base in the literature, for example, Goldszmidt and Pearl (1996), Giordano *et al*. (2015), Casini *et al*. (2019). To distinguish two different notions of consistency that commonly occur and are both used in this paper, we call one notion of consistency *strong consistency* and the other notion *weak consistency*, as suggested in Haldimann *et al*. (2023).

Definition 7 (Haldimann et al., 2023). *A belief base Δ is called strongly consistent if there exists at least one ranking function κ with $\kappa \models \Delta$ and $\kappa^{-1}(\infty) = \emptyset$. A belief base Δ is weakly consistent if there is a ranking function κ with $\kappa \models \Delta$.*

Thus, Δ is strongly consistent if there is at least one ranking function modelling Δ that considers all worlds feasible. This notion of consistency is used in many approaches, for example, Goldszmidt and Pearl (1996). The notion of weak consistency is equivalent to the more relaxed notion of consistency that is used in, for example, Giordano *et al*. (2015), Casini *et al*. (2019). Trivially, strong consistency implies weak consistency.

Example 8. *Let $\Sigma = \{a, b, c, d\}$. The belief base $\Delta_1 = \{(\bot|\top)\}$ is not weakly consistent. If there were any ranking function κ with $\kappa \models \Delta_1$ then there would be a world ω such that $\kappa(\omega) = 0$ and therefore $\kappa(\top) = 0$. For κ to model $(\bot|\top)$, we need $\kappa(\top \wedge \bot) < \kappa(\top \wedge \top) = 0$, which is clearly impossible. $\Delta_2 = \{(\bot|a), (\overline{b}|\overline{a}), (b|\overline{a})\}$ is also not weakly consistent. The conditional $(\bot|a)$ enforces that for every ranking function κ with $\kappa \models \Delta_2$ and any model ω of a we have $\kappa(a) = \infty$. The conditionals $(\overline{b}|\overline{a}), (b|\overline{a})$ in combination enforce that the rank of any model of $\overline{a}$ must have rank infinity. Because every ranking function must assign rank 0 to at least one world, there is no ranking function modelling Δ_2. Because Δ_1 and Δ_2 are not weakly consistent, they are not strongly consistent.*

The belief bases $\Delta_3 = \{(\overline{b}|\overline{a}), (b|\overline{a})\}$ and $\Delta_4 = \{(\bot|a)\}$, both subsets of Δ_2, are weakly consistent but not strongly consistent. Δ_3 requires all ranking functions modelling it to assign rank ∞ to models of $\overline{a}$, and Δ_4 requires all ranking functions modelling it to assign rank ∞ to models of a.

The belief base $\Delta_5 = \{(b|a), (d|c)\}$ is strongly consistent and thus also weakly consistent.

Lemma 9. *For every weakly consistent belief base Δ there is an $\omega \in \Omega$ such that ω does not falsify any conditional in Δ.*

Proof. Because Δ is weakly consistent, there is a ranking function κ with $\kappa \models \Delta$. Let $\omega \in \kappa^{-1}(0)$. Towards a contradiction, assume that there is a $(B|A) \in \Delta$ that is falsified by ω, that is, $\omega \models A\overline{B}$. For κ to accept $(B|A)$ it must be either $\kappa(A) = \infty$ or $\kappa(AB) < \kappa(A\overline{B})$. Because $\omega \models A$ and $\kappa(\omega) = 0$ we have $\kappa(A) \neq \infty$. Because $\kappa(A\overline{B}) \leqslant 0$ and there are no ranks below 0 the condition $\kappa(AB) < \kappa(A\overline{B})$ does not hold. This is a contradiction; hence, ω does not falsify any conditional in Δ. □

Lemma 10. *Let Δ be a belief base. If there is an $\omega \in \Omega$ such that ω does not falsify any conditional in Δ, then Δ is weakly consistent.*

Proof. Assume there is a world ω^* that does not falsify any conditional in Δ. Then

$$\kappa(\omega) = \begin{cases} 0 & \text{if } \omega = \omega^* \\ \infty & \text{otherwise} \end{cases}$$

is a model of Δ. Therefore, Δ is weakly consistent. □

An interpretation ω falsifies a conditional $(B|A)$ iff it is not a model of its material implication counterpart $A \rightarrow B$, which is equivalent to $\neg A \vee B$. Therefore, Lemmas 9 and 10 imply that checking a belief base for weak consistency can be reduced to a single SAT check.

Proposition 11. *A belief base Δ is weakly consistent if*

$$\bigwedge_{(B_i|A_i) \in \Delta} \neg A \vee B$$

is satisfiable.

The original definition of an inductive inference operator in Kern-Isberner *et al*. (2020) implicitly assumes all belief bases to be strongly consistent. However, the definition can be extended to cover all belief bases. Some inductive inference operators known from literature are defined only for strongly consistent belief bases, while other operators are defined for all belief bases. In particular, inference operators of the second type are able to draw inferences from all weakly consistent belief bases. For example, system Z is an inductive inference operator that is defined based on the Z-partition of a belief base. It was first defined for strongly consistent belief bases (Pearl, 1990). Then an extended version of system Z was introduced (Goldszmidt & Pearl, 1990) that also covers weakly consistent belief bases and that was shown to be equivalent to *rational closure* (Lehmann, 1989) in Goldszmidt and Pearl (1990).

Definition 12 ((extended) Z-partition). *A conditional $(B|A)$ is tolerated by $\Delta = \{(B_i|A_i) \mid i = 1, \dots, n\}$ if there is a world $\omega \in \Omega$ such that ω verifies $(B|A)$ and ω does not falsify any conditional in Δ, that is, $\omega \models AB$ and $\omega \models \bigwedge_{i=1}^{n} (\overline{A_i} \vee B_i)$.*

The (extended) Z-partition $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$ of a belief base Δ is the ordered partition of Δ constructed by letting Δ^i be the inclusion maximal subset of $\bigcup_{j=i}^{n} \Delta^j$ that is tolerated by $\bigcup_{j=i}^{n} \Delta^j$ until $\Delta^{k+1} = \emptyset$. The set Δ^∞ is the remaining set of conditionals containing no conditional tolerated by Δ^∞.

Because the Δ^i are chosen inclusion-maximal, the Z-partition is unique (Pearl, 1990).

Definition 13 ((extended) system Z). *Let Δ be a belief base with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$. If Δ is not weakly consistent, let $A \mathrel{|\!\!\sim}^z_\Delta B$ for any $A, B \in \mathcal{L}$. Otherwise, the (extended) Z-ranking function κ^z_Δ is defined as follows: For $\omega \in \Omega$, if a conditional in Δ^∞ is applicable to ω define $\kappa^z_\Delta(\omega) = \infty$. If not, let Δ^j be the last element in $EZP(\Delta)$ that contains a conditional falsified by ω. Then let $\kappa^z_\Delta(\omega) = j + 1$. If ω does not falsify any conditional in Δ, then let $\kappa^z_\Delta(\omega) = 0$. (Extended) system Z maps Δ to the inference relation $\mathrel{|\!\!\sim}^z_\Delta$ induced by κ^z_Δ.*

For weakly consistent belief bases Δ the OCF κ^z_Δ is a model of Δ. For strongly consistent belief bases extended system Z coincides with system Z in Pearl (1990), Goldszmidt and Pearl (1996).

Lemma 14 (Haldimann *et al*., 2023). *For a weakly consistent belief base Δ and a formula A we have $\kappa^z_\Delta(A) = \infty$ iff $A \mathrel{|\!\!\sim}^p_\Delta \bot$.*

Lemma 15 (Haldimann *et al*., 2023). *Let Δ be with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$. A world $\omega \in \Omega$ falsifies a conditional in Δ^∞ iff a conditional in Δ^∞ is applicable in ω.*

Proof. Direction $\Rightarrow$: Assume that ω falsifies a conditional in Δ^∞. Then this conditional is applicable for ω.

Direction $\Leftarrow$: Assume that at least one conditional $(B|A) \in \Delta^\infty$ is applicable in ω. There are two possible cases: Either ω falsifies one of the other conditionals in Δ^∞ or not. In the first case the lemma holds. In the second case, towards a contradiction, we assume that ω does not falsify $(B|A)$. If $(B|A)$ is applicable in ω and ω does not falsify $(B|A)$ then ω must verify $(B|A)$. That implies that $(B|A)$ is tolerated by Δ^∞ which contradicts the construction of $EZP(\Delta)$. □

It is well-known that the construction of the extended Z-partition $EZP(\Delta)$ is successful with $\Delta^\infty = \emptyset$ iff Δ is strongly consistent (Goldszmidt & Pearl, 1996). We can also use the extended Z-partition to check for weak consistency. The following proposition summarizes the relations between $EZP(\Delta)$ and the consistency of Δ.

Proposition 16. *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a belief base with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$.*

1. *Δ is strongly consistent iff $\Delta^\infty = \emptyset$.*
2. *Δ is weakly consistent iff $\Delta^\infty \neq \Delta$ or $A_1 \vee \dots \vee A_n \not\equiv \top$.*
3. *Δ is not weakly consistent iff $\Delta^\infty = \Delta$ and $A_1 \vee \dots \vee A_n \equiv \top$.*

Continuing Example 8, for the not weakly consistent Δ_2 we have $EZP(\Delta_2) = (\Delta_2^\infty)$ with $\Delta_2^\infty = \Delta$ and $a \vee \overline{a} \vee \overline{a} \equiv \top$. For the weakly consistent Δ_3 we have $EZP(\Delta_3) = (\Delta_3^\infty)$ with $\Delta_3^\infty = \Delta$ but $a \not\equiv \top$. For the strongly consistent Δ_3 we have $EZP(\Delta_4) = (\Delta_4^0)$ with $\Delta_4^0 = \Delta$ and $\Delta_4^\infty = \emptyset$.

The extended Z-partition $EZP(\Delta)$ can be computed with the algorithm provided by Pearl (1990). Having $EZP(\Delta)$ at hand, κ_Δ^z can be computed directly along the lines of Definition 13, leading to an implementation of system Z inference induced by κ_Δ^z for every weakly consistent belief base Δ.

For p-entailment with respect to strongly consistent belief bases, it is well known that the relation $A \mid\!\sim^p_\Delta B$ holds iff $\Delta \cup \{(\overline{B}|A)\}$ is not strongly consistent (Goldszmidt & Pearl, 1996). Obviously, this characterization of p-entailment is not applicable for belief bases that are only weakly consistent. For p-entailment with respect to a weakly consistent belief base Δ, Lehmann and Magidor (1992), p. 41, say that Dix pointed out to them how rational closure, and thus extended system Z, yields a method for deciding whether $A \mid\!\sim^p_\Delta B$ holds. We state this observation in the following proposition.

Proposition 17 (Lehmann & Magidor, 1992, p. 41, adapted). *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a weakly consistent belief base, and let A, B be formulas. Then the following holds:*

$$A \mid\!\sim^p_\Delta B \quad \textit{iff} \quad A \mid\!\sim^z_{\Delta'} \bot \quad \textit{where} \quad \Delta' = \Delta \cup \{(\overline{B}|A)\} \tag{2}$$

An immediate consequence of Proposition 17 is the following.

Proposition 18. *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a weakly consistent belief base, and let A, B be formulas. Then the following holds:*

$$A \mid\!\sim^p_\Delta B \quad \textit{iff} \quad \kappa^z_{\Delta'}(A) = \infty \quad \textit{where} \quad \Delta' = \Delta \cup \{(\overline{B}|A)\} \tag{3}$$

Proof. The claim follows from Proposition 17 because for every ranking function κ, the relation $A \mid\!\sim_\kappa \bot$ iff $\kappa(A) = \infty$ holds according to Equation (1). □

Hence, an implementation of (extended) system Z and its underlying ranking function can be used to obtain an implementation of p-entailment, and thus of system P, for every weakly consistent belief base.

5. Generalizing c-representations

c-Representations are a special type of ranking model of a belief base. They are obtained from natural number impacts associated with the conditionals in the belief base that can be seen as penalty points for worlds falsifying the corresponding conditional.

Definition 19 (c-representation, Kern-Isberner, 2001, 2004). *A c-representation of a belief base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ over Σ is a ranking function $\kappa_{\vec{\eta}}$ constructed from integers $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, also called impacts, with $\eta_i \in \mathbb{N}_0$, $i \in \{1, \ldots, n\}$ assigned to each conditional $(B_i|A_i)$ such that $\kappa_{\vec{\eta}}$ accepts Δ and is given by:*

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i. \tag{4}$$

We will denote the set of all c-representations of Δ by $Mod^c_\Sigma(\Delta)$.

Note that the impact η_i assigned to the conditional $(B_i|A_i)$ decreases the plausibility of a world ω if ω falsifies $(B_i|A_i)$, and that the rank of ω under the c-representation $\kappa_{\vec{\eta}}$ induced by the impact vector $\vec{\eta}$ is the sum of the impacts assigned to the conditionals which are falsified by ω.

A belief base Δ that is not strongly consistent has no c-representation: by Definition 19, a c-representation of Δ is a finite ranking function modelling Δ; if Δ is not strongly consistent, such a ranking function cannot exist. For belief bases that are only weakly consistent, we need a more general definition of c-representations. A ranking function that is a model of a weakly but not strongly consistent belief base must assign rank ∞ to some worlds. To achieve this while keeping a construction of

Table 1. *Verification (v) and falsification (f) of the conditionals in Δ from Example 21 and their corresponding impacts. The ranking function $\kappa_{\vec{\eta}}$ induced by the impacts $\vec{\eta} = (\eta_1, \eta_2, \eta_3) = (\infty, 1, \infty)$ is an extended c-representation for Δ*

ω	$(b\|p)$	$(f\|b)$	$(\overline{b}\|p)$	impact on ω	$\kappa_{\vec{\eta}}(\omega)$
bpf	v	v	f	η_3	∞
$bp\overline{f}$	v	f	f	$\eta_2 + \eta_3$	∞
$b\overline{p}f$	–	v	–	0	0
$b\overline{p}\overline{f}$	–	f	–	η_2	1
$\overline{b}pf$	f	–	v	η_1	∞
$\overline{b}p\overline{f}$	f	–	v	η_1	∞
$\overline{b}\overline{p}f$	–	–	–	0	0
$\overline{b}\overline{p}\overline{f}$	–	–	–	0	0
impacts:	η_1	η_2	η_3		
$\vec{\eta}$	∞	1	∞		

c-representations similar to the one given in (4), we extend the definition of c-representations to allow infinite impacts.

Definition 20 (extended c-representation). *An extended c-representation of a belief base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ over Σ is a ranking function $\kappa_{\vec{\eta}}$ constructed from impacts $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ with $\eta_i \in \mathbb{N}_0 \cup \{\infty\}, i \in \{1, \ldots, n\}$ assigned to each conditional $(B_i|A_i)$ such that $\kappa_{\vec{\eta}}$ accepts Δ and is given by*

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{5}$$

We will denote the set of all extended c-representations of Δ by $Mod^{ec}_{\Sigma}(\Delta)$.

Example 21. *Let $\Sigma = \{b, p, f\}$ and $\Delta = \{(b|p), (f|b), (\overline{b}|p)\}$. Note that Δ is weakly consistent but not strongly consistent. The OCF $\kappa_{\vec{\eta}}$ displayed in Table 1 is an extended c-representation of Δ induced by the impacts $\vec{\eta} = (\infty, 1, \infty)$.*

Every c-representation of a strongly consistent belief base Δ is obviously an extended c-representation of Δ, and every weakly consistent belief base has at least one extended c-representation.

Proposition 22. *Let Δ be a strongly consistent belief base. Every c-representation $\kappa_{\vec{\eta}}$ of Δ is an extended c-representation of Δ.*

Proposition 23. *Let Δ be a weakly consistent belief base. Then $\kappa_{\vec{\eta}}$ with $\vec{\eta} = (\infty, \ldots, \infty)$ is an extended c-representation of Δ.*

Proof. Because Δ is weakly consistent, there is at least one world $\omega \in \Omega_\Sigma$ that does not falsify any of the conditionals (see Lemma 9). This implies $\kappa_{\vec{\eta}}(\omega) = 0$. Thus, $\kappa_{\vec{\eta}}$ is a ranking function.

For every $(B|A) \in \Delta$ it holds that $\kappa_{\vec{\eta}}(A\overline{B}) = \infty$ because every model of $A\overline{B}$ falsifies the conditional $(B|A)$ with impact ∞. For $\kappa_{\vec{\eta}}(AB)$ we have either (1) $\kappa_{\vec{\eta}}(AB) = 0$ or (2) $\kappa_{\vec{\eta}}(AB) = \infty$. In case (1) we have $\kappa_{\vec{\eta}}(AB) = 0 < \infty = \kappa_{\vec{\eta}}(A\overline{B})$. In case (2) we have $\kappa_{\vec{\eta}}(AB) = \infty$ and $\kappa_{\vec{\eta}}(A\overline{B}) = \infty$ and therefore $\kappa_{\vec{\eta}}(A) = \infty$ because $\kappa_{\vec{\eta}}(A) = \min\{\kappa_{\vec{\eta}}(AB), \kappa_{\vec{\eta}}(A\overline{B})\}$. In both cases, $\kappa_{\vec{\eta}}$ accepts $(B|A)$. Thus, $\kappa_{\vec{\eta}} \models \Delta$. □

Proposition 23 also illustrates that in extended c-representations worlds may have rank infinity without the belief base requiring this. In an extended c-representation of Δ only those worlds need to have rank infinity that have rank infinity in the z-ranking κ^z_Δ of Δ. This is shown in the following two lemmas.

Lemma 24. *Let Δ be a weakly consistent belief base. If $\kappa^z_\Delta(\omega) = \infty$ for a world ω, then $\kappa_{\vec{\eta}}(\omega) = \infty$ for all extended c-representations $\kappa_{\vec{\eta}}$ of Δ.*

Proof. Assume that $\kappa^z_\Delta(\omega) = \infty$. Let $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$ be the extended Z-partition of Δ. By definition of κ^z_Δ there exists a conditional $(B|A) \in \Delta^\infty$ s.t. $\omega \models A$. Because $(B|A) \in \Delta^\infty$ the conditional $(B|A)$ is not tolerated by Δ^∞, so there is a conditional $(B'|A') \in \Delta^\infty$ that is falsified by ω (this can be $(B|A)$ again).

Towards a contradiction assume that there is a c-representation $\kappa_{\vec{\eta}}$ of Δ with $\kappa_{\vec{\eta}}(\omega) < \infty$. As $\kappa_{\vec{\eta}}$ models Δ and thus also $(B'|A')$ there must be a world ω^1 that verifies $(B'|A')$ and satisfies $\kappa_{\vec{\eta}}(\omega^1) < \kappa_{\vec{\eta}}(\omega)$. With the same argumentation there must be another conditional $(B^1|A^1) \in \Delta^\infty$ that is falsified by ω^1, and another world ω_2 that verifies $(B^1|A^1)$ and satisfies $\kappa_{\vec{\eta}}(\omega^2) < \kappa_{\vec{\eta}}(\omega^1)$. Repeating this argumentation, we obtain an infinite chain of worlds $\omega_1, \omega_2, \ldots$ s.t. $\kappa_{\vec{\eta}}(\omega_1) > \kappa_{\vec{\eta}}(\omega_2) > \ldots$. But as there are only finitely many worlds (and also because there are only finitely many ranks below $\kappa_{\vec{\eta}}(\omega_1)$) such a chain cannot exist. Contradiction. □

Lemma 25. *Let Δ be a weakly consistent belief base. There is a c-representation $\kappa_{\vec{\eta}}$ of Δ with $\kappa_{\vec{\eta}}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$.*

Proof. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a weakly consistent belief base. Let $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$ be the extended Z-partition of Δ. Construct an impact vector $\vec{\eta}$ for Δ as follows. Let $\mu^0 = 1$ and $\mu^j = |\Delta^0 \cup \cdots \cup \Delta^{j-1}| \cdot \mu^{j-1} + 1$ for $j = 1, \ldots, m$. For $(B_i|A_i)$ with $(B_i|A_i) \in \Delta^j$ let $\eta_i = \mu^j$ for $j < \infty$ and $\eta_j = \infty$ for $j = \infty$. By construction, for worlds ω that do not falsify a conditional from $\Delta^j \cup \cdots \cup \Delta^m \cup \Delta^\infty$ we have $\kappa_{\vec{\eta}}(\omega) < \mu^j$.

$\kappa_{\vec{\eta}}$ is a c-representation of Δ: Let $(B_i|A_i)$ be any conditional in Δ. If $(B_i|A_i) \in \Delta^\infty$ then $\kappa^z_\Delta(A_i) = \infty$ by the definition of κ^z_Δ which implies with Proposition 26 that $\kappa_{\vec{\eta}}(A_i) = \infty$ and therefore $\kappa_{\vec{\eta}} \models (B_i|A_i)$. Otherwise, we have $(B_i|A_i) \in \Delta^j$ with $j < \infty$. Then for any world ω' falsifying $(B_i|A_i)$ we have $\kappa_{\vec{\eta}}(\omega) > \mu^j$; hence $\kappa_{\vec{\eta}}(A_i\overline{B_i}) \geqslant \mu_j$. Because $(B_i|A_i) \in \Delta^j$, there is a world ω' that verifies $(B_i|A_i)$ and does not falsify a conditional in $\Delta^j \cup \cdots \cup \Delta^m \cup \Delta^\infty$. Therefore, $\kappa_{\vec{\eta}}(A_iB_i) < \mu_j$. Thus, $\kappa_{\vec{\eta}}(A_iB_i) < \mu_j \leqslant \kappa_{\vec{\eta}}(A_i\overline{B_i})$ and $\kappa_{\vec{\eta}} \models (B_i|A_i)$.

Furthermore, it holds that $\kappa_{\vec{\eta}}(\omega) = \infty$ iff ω falsifies a conditional in Δ^∞. Therefore, $\kappa_{\vec{\eta}}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$. □

Lemmas 24 and 25 can be summarized by the following proposition.

Proposition 26. *Let Δ be a weakly consistent belief base. If $\kappa^z_\Delta(\omega) = \infty$ for a world ω, then $\kappa_{\vec{\eta}}(\omega) = \infty$ for all extended c-representations $\kappa_{\vec{\eta}}$ of Δ.*

Moreover, there is an extended c-representation $\kappa_{\vec{\eta}}$ of Δ with $\kappa_{\vec{\eta}}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$.

As a consequence of this proposition, for a weakly consistent Δ, there is an extended c-representation $\kappa_{\vec{\eta}}$ such that $\kappa_{\vec{\eta}}(\omega) < \infty$ iff $\kappa^z_\Delta(\omega) < \infty$. Using Lemma 14, we have $\kappa_{\vec{\eta}}(\omega) < \infty$ iff ω does not entail $\bot$ with p-entailment.

Lemma 27. *Let Δ be a weakly consistent belief base. There is an extended c-representation $\kappa_{\vec{\eta}}$ of Δ such that for all $\omega \in \Omega$ we have $\kappa_{\vec{\eta}}(\omega) < \infty$ iff $\omega \not\models^p_\Delta \bot$, where the world ω is considered as a formula on the right side of the 'iff'.*

Another consequence of Proposition 26 is the following.

Proposition 28. *Let Δ be a belief base with $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$, and let $\omega \in \Omega$. We have that $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$ iff $\omega \models A$ for some $(B|A) \in \Delta^\infty$.*

Proof. **Direction** $\Rightarrow$ If $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$, then there is no $\kappa_{\vec{\eta}} \in Mod^{ec}_\Delta$ with $\kappa_{\vec{\eta}}(\omega) < \infty$. With Proposition 26 this implies $\kappa^z_\Delta(\omega) = \infty$. By Definition 13, this is the case if a conditional in Δ^∞ is applicable for ω.

Direction $\Leftarrow$ Assume $\omega \models A$ for some $(B|A) \in \Delta^\infty$. Then $\kappa^z_\Delta(\omega) = \infty$ and with Proposition 26 we have $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$. □

When general c-representations were introduced in Kern-Isberner (2001), they were shown to be *conditionally indifferent* with respect to the inducing knowledge base. Very briefly, conditional indifference is an invariance property for models of a knowledge base which claims that possible worlds should be assigned the same semantic value if they behave the same with respect to the conditionals in the knowledge base, that is, if they verify and falsify exactly the same conditionals. Equivalently, this means that any difference in the semantic evaluation of two worlds can be explained by differences in the conditional-logical evaluation of the worlds with respect to the knowledge base. This links the numerical values assigned to worlds clearly to the logical structure provided by the conditionals in the base.

We adapt a simplified version of conditional indifference to the setting of inducing ranking models from weakly consistent knowledge bases on which we focus in this paper.

Definition 29 (conditional indifference, adapted from Kern-Isberner, 2001). *Let Δ be a conditional belief base and let $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ be a ranking function. Then κ is conditionally indifferent with respect to Δ, if the following two conditions hold:*

1. *If $\kappa(\omega) = \infty$, then there is a conditional $(B|A) \in \Delta$ that is either falsified in or not applicable to ω, and $\kappa(\omega') = \infty$ for every $\omega' \in \Omega$ that behaves the same with respect to $(B|A)$ as ω, that is, $(B|A)$ also falsifies resp. is not applicable to ω'.*
2. *$\kappa(\omega) = \kappa(\omega')$ for every $\omega, \omega' \in \Omega$ that verify and falsify exactly the same conditionals in Δ.*

The following proposition shows that also extended c-representations of weakly consistent knowledge bases satisfy conditional indifference.

Proposition 30. *Let Δ be a conditional belief base and let $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ be an extended c-representation of Δ. Then the following two conditions hold:*

1. *If $\kappa(\omega) = \infty$, then there is a conditional $(B|A) \in \Delta$ that is either falsified in or not applicable to ω, and $\kappa(\omega') = \infty$ for every $\omega' \in \Omega$ that behaves the same with respect to $(B|A)$ as ω, that is, $(B|A)$ also falsifies resp. is not applicable to ω'.*
2. *$\kappa(\omega) = \kappa(\omega')$ for every $\omega, \omega' \in \Omega$ that verify and falsify exactly the same conditionals in Δ.*

Proof. **Ad 1:** Let ω be a world with $\kappa(\omega) = \infty$. Because $\kappa(\omega)$ is the sum of the impacts of the conditionals falsified by ω and Δ is finite, ω must falsify at least one conditional $(B_i|A_i)$ with impact $\eta_i = \infty$. This $(B_i|A_i)$ satisfies the first part of condition (1). Moreover, all worlds ω' that also falsify $(B_i|A_i)$ will have rank $\kappa(\omega') = \infty$, too. Therefore, also the second part of condition (1) is satisfied by $(B_i|A_i)$, and the condition holds.

Ad 2: The rank $\kappa(\omega)$ of a world ω only depends on which conditionals in Δ the world falsifies. Therefore, if two worlds ω, ω' verify and falsify the same conditionals in Δ they will have the same rank. □

With extended c-representations we can now define extended c-inference.

6. Extending c-inference

c-Inference (Beierle *et al*., 2016a; 2018) is an inference operator taking all c-representations of a belief base Δ into account. It was originally defined for strongly consistent belief bases.

Definition 31 (c-inference, $\mid\!\sim^c_\Delta$ Beierle et al., 2016a). *Let Δ be a strongly consistent belief base and let A, B be formulas. B is a c-inference from A in the context of Δ, denoted by $A \mid\!\sim^c_\Delta B$, iff $A \mid\!\sim_\kappa B$ holds for all c-representations κ of Δ.*

Now we use extended c-representations to extend c-inference for belief bases that may be only weakly consistent.

Definition 32 (extended c-inference, $\mid\!\sim^{ec}_{\Delta}$). *Let Δ be a belief base and let $A, B \in \mathcal{L}$. Then B is an extended c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{ec}_{\Delta} B$, iff $A \mid\!\sim_{\kappa} B$ holds for all extended c-representations κ of Δ.*

First, let us verify that extended c-inference is indeed an inductive inference operator that coincides with c-inference for strongly consistent belief bases.

Proposition 33. *Extended c-inference is an inductive inference operator, that is, it satisfies (DI) and (TV).*

Proof. We need to show that extended c-inference satisfies (DI) and (TV). (DI) is trivial: Every c-representation of Δ accepts the conditionals in Δ by definition. Therefore, $A \mid\!\sim^{ec}_{\Delta} B$ for every $(B|A) \in \Delta$. (TV) is also clear: For $\Delta = \emptyset$, the only c-representation is κ_u with $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$. In this case, κ_u accepts only conditionals $(B|A)$ with $A\overline{B} = \bot$, which are conditionals with $A \models B$. □

Proposition 34. *For strongly consistent belief bases, extended c-inference coincides with normal c-inference.*

Proof. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a strongly consistent belief base and $C, D \in \mathcal{L}$. We need to show that $C \mid\!\sim^{ec}_{\Delta} D$ iff $C \mid\!\sim^{c}_{\Delta} D$.

Direction $\Rightarrow$: Let $C \mid\!\sim^{ec}_{\Delta} D$, that is, every extended c-representation models $(D|C)$. As every c-representation is an extended c-representation (Proposition 22), every c-representation models $(D|C)$. Thus, $D \mid\!\sim^{c}_{kb} C$.

Direction $\Leftarrow$: Let $C \mid\!\sim^{c}_{\Delta} D$, that is, every c-representation models $(D|C)$. We need to show that any extended c-representation $\kappa_{\vec{\eta}}$ models $(D|C)$. If $\vec{\eta}$ contains only finite values, it is a c-representation and thus models $(D|C)$ by assumption.

Assume that $\vec{\eta}$ contains infinite entries. Let $OP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$ be the extended tolerance partition of Δ. Because Δ is strongly consistent, we have $\Delta^\infty = \emptyset$. Let $fin(\vec{\eta}) = \{\eta_i \mid i \in \{0, \ldots, n\}, \eta_i < \infty\}$ be the set of finite values in impact vector $\vec{\eta}$ and $f_0 = 1 + |fin(\vec{\eta})| \cdot \max(fin(\vec{\eta}))$. Now construct $\vec{\eta}^f$ from $\vec{\eta}$ as follows. For $(B_i|A_i) \in \Delta^0$ with $\eta_i = \infty$ let $\eta^f_i = f_0$. Let $f_1 = (f_0 + 1) \cdot |\{(B_i|A_i) \in \Delta^0 \mid \eta_i = \infty\}|$. For $(B_i|A_i) \in \Delta^1$ with $\eta_i = \infty$ let $\eta^f_i = f_1$. Let $f_2 = (f_1 + 1) \cdot \left|\{(B_i|A_i) \in \Delta^1 \mid \eta_i = \infty\}\right|$. For $(B_i|A_i) \in \Delta^1$ with $\eta_i = \infty$ let $\eta^f_i = f_2$; and so on. By construction the sum of the impacts in $fin(\vec{\eta})$ is less than f_0 and the sum of the impacts of the conditionals in $\Delta^0 \cup \cdots \cup \Delta^j$ is less than f_j for $j = 0, \ldots, m$.

Let $\kappa^f = \kappa_{\vec{\eta}^f}$. Now verify that:

1. κ^f is a c-representation of Δ. For this we need to check that κ^f models all conditionals in Δ.
2. $\mid\!\sim_{\kappa^f} \subseteq \mid\!\sim_{\kappa_{\vec{\eta}}}$, that is, every inference in $\mid\!\sim_{\kappa^f}$ is also an inference in $\mid\!\sim_{\kappa_{\vec{\eta}}}$.

From (1) it follows that κ^f is a model of $(D|C)$ because $C \mid\!\sim^{c}_{\Delta} D$. With (2.) it follows that $(D|C)$ is modelled by $\kappa_{\vec{\eta}}$.

Ad (1): Let $(B_i|A_i) \in \Delta$. We distinguish three cases.

Case 1: $\kappa_{\vec{\eta}}(A_iB_i) < \kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$

In this case $\kappa^f(A_iB_i) < \kappa^f(A_i\overline{B_i}) < f_0$ and therefore $\kappa^f \models (B_i|A_i)$.

Case 2: $\kappa_{\vec{\eta}}(A_iB_i) < \infty$ and $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$

In this case $\kappa^f(A_iB_i) < f_0 < \kappa^f(A_i\overline{B_i})$ and therefore $\kappa^f \models (B_i|A_i)$.

Case 3: $\kappa_{\vec{\eta}}(A_iB_i) = \infty$ and $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$

Assume that $(B_i|A_i)$ is in Δ^j. Then there is a world ω s.t. $\omega \models A_iB_i$ and ω falsifies no conditional in $\Delta^0 \cup \cdots \cup \Delta^j$. Therefore, $\kappa^f(\omega) < f_j$ and thus $\kappa^f(A_iB_i) < f_j$. Any model of $A_i\overline{B_i}$ falsifies $(B_i|A_i)$, therefore, $\kappa^f(A_i\overline{B_i}) > f_j$. Thus, we have $\kappa^f(A_iB_i) < f_j < \kappa^f(A_i\overline{B_i})$ and therefore $\kappa^f \models (B_i|A_i)$.

Ad (2): Assume that $X \mid\!\sim_{\kappa^f} Y$. There are two cases.

Case 1: $\kappa^f(X\overline{Y}) < f_0$

In this case $\kappa^f(XY) < \kappa^f(X\overline{Y}) < f_0$ and therefore $\kappa_{\vec{\eta}}(XY) < \kappa_{\vec{\eta}}(X\overline{Y}) < \infty$. Hence, $X \mid\!\sim_{\kappa_{\vec{\eta}}} Y$.

Case 2: $\kappa^f(X\overline{Y}) \geqslant f_0$

In this case $\kappa_{\vec{\eta}}(X\overline{Y}) = \infty$ and therefore $X \mid\!\sim_{\kappa_{\vec{\eta}}} Y$. □

Because extended c-inference is defined as skeptical inference over a set of ranking functions it is also a preferential inference operator.

Proposition 35. *Extended c-inference is preferential, that is, it satisfies system P.*

Proof. Every ranking function, and thus every extended c-representation, induces a preferential inference relation. The intersection of two preferential inference relations is again preferential. As extended c-inference is the intersection of the inference relations induced by each extended c-representation, extended c-inference is preferential. □

Proposition 35 implies that extended c-inference captures p-entailment, that is, if $A \mid\!\sim^{p}_{\Delta} B$, then $A \mid\!\sim^{ec}_{\Delta} B$. Furthermore, extended c-inference coincides with p-entailment on entailments of the form $A \mid\!\sim \bot$, which can be seen as representations of 'strict' beliefs (i.e., A is completely unfeasible).

Proposition 36. *Extended c-inference satisfies (Classic Preservation).*

Proof. We need to show that $A \mid\!\sim^{ec}_{\Delta} \bot$ iff $A \mid\!\sim^{p}_{\Delta} \bot$. Using Lemma 14, it is sufficient to show that $A \mid\!\sim^{ec}_{\Delta} \bot$ iff $\kappa^{z}(A) = \infty$.

Direction $\Leftarrow$: Let $\kappa^{z}_{\Delta}(A) = \infty$. Then Proposition 26 states that $\kappa_{\vec{\eta}}(A) = \infty$ for every extended c-representation $\kappa_{\vec{\eta}}(A)$ of Δ. Thus, $A \mid\!\sim^{ec}_{\Delta} \bot$.

Direction $\Rightarrow$: Let $A \mid\!\sim^{ec}_{\Delta} \bot$, that is, there is no extended c-representation $\kappa_{\vec{\eta}}$ of Δ s.t. $\kappa_{\vec{\eta}}(A) < \infty$. By Proposition 26, we have $\kappa^{z}_{\Delta}(A) = \infty$. □

Besides *Cautious Monotony (CM)* ensured by system P, other postulates capturing versions of monotony have been introduced. One of these is *Rational Monotony* (Kraus *et al.*, 1990) which in combination with system P characterizes the class of inference relations that can be induced by ranking functions (Lehmann & Magidor, 1992).

(**RM**) *Rational Monotony* $\quad A \mid\!\sim B$ *and* $A \not\mid\!\sim \overline{C}$ *imply* $AC \mid\!\sim B$

Another monotony postulate is *Semi Monotony* (Reiter, 1980; Goldszmidt & Pearl, 1996), which states that the set of possible inferences only increases if conditionals are added to the belief base.

(**SM**) *Semi Monotony* $\quad \Delta \subseteq \Delta'$ *and* $A \mid\!\sim_{\Delta} B$ *imply* $A \mid\!\sim_{\Delta'} B$

Extended c-inference does not satisfy (RM) as c-inference already violates (RM) (Beierle *et al.*, 2019a). Analogously, extended c-inference does not satisfy (SM) (Kutsch, 2021). However, we can show that extended c-inference satisfies weaker versions of these two postulates: *weak Rational Monotony* (Rott, 2001) and *weak Semi Monotony* (Wilhelm *et al.*, 2024).

(**wRM**) *weak Rational Monotony* $\quad \top \mid\!\sim B$ *and* $\top \mid\!\sim \overline{A}$ *imply* $A \mid\!\sim B$
(**wSM**) *weak Semi Monotony* $\quad \Delta \subseteq \Delta'$ *and* $\top \mid\!\sim_{\Delta} B$ *imply* $\top \mid\!\sim_{\Delta'} B$

c-Inference satisfies weak rational monotony (Beierle *et al.*, 2019a), and this also holds for the extended version.

Proposition 37. *Extended c-inference satisfies (wRM).*

Proof. Let Δ be a belief base with $\top \mid\!\sim^{ec}_{\Delta} B$ and $\top \not\mid\!\sim^{ec}_{\Delta} \overline{A}$. If Δ is not weakly consistent, $A \mid\!\sim^{ec}_{\Delta} B$ holds trivially. For the remainder of the proof assume that Δ is weakly consistent. Let κ be any extended c-representation of Δ, and let $\Omega_{zero} = \kappa^{-1}(0)$ be the set of worlds with rank 0 with respect to κ. Then, $\top \mid\!\sim^{ec}_{\Delta} B$ implies that $\kappa(B) < \kappa(\overline{B})$ and therefore that for every $\omega \in \Omega_{zero}$ we have $\omega \models B$. Because $\top \not\mid\!\sim^{ec}_{\Delta} \overline{A}$ there is at least one $\omega \in \Omega_{zero}$ with $\omega \models A$. This ω is a model of AB implying that $\kappa(AB) = 0$. We also have that $\kappa(A\overline{B}) > 0$ because there are no models of B with rank 0. In summary, we have $\kappa(AB) = 0 < \kappa(A\overline{B})$ implying that $A \mid\!\sim_{\kappa} B$. Because the c-representation κ was chosen arbitrarily, we have $A \mid\!\sim^{ec}_{\Delta} B$. □

While c-Inference and its extended version fail to satisfy semi monotony (Kutsch, 2021), both satisfy the only recently introduced (Wilhelm *et al.*, 2024) weaker notion of it.

Proposition 38. *c-Inference and extended c-inference satisfy (wSM).*

Proof. We first show that extended c-inference satisfies (wSM). Let Δ, Δ' be belief bases with $\Delta \subseteq \Delta'$ and $\top \mid\!\sim^{ec}_{\Delta} B$. If Δ' is not weakly consistent, $\top \mid\!\sim^{ec}_{\Delta'} B$ holds trivially. For the remainder of the proof assume that Δ' and therefore also Δ is weakly consistent. Let κ' be an extended c-representation of Δ', and let $\Omega_{zero} = \kappa'^{-1}(0)$ be the set of worlds with rank 0 with respect to κ'. Let ω be any world in Ω_{zero}. Because $\kappa' \models \Delta'$, the world ω cannot falsify any conditional in Δ'. Thus, ω also does not falsify any conditional in Δ. Let κ be an extended c-representation of Δ. Because ω does not falsify any conditional in Δ, we have $\kappa(\omega) = 0$. Since $\top \mid\!\sim^{ec}_{\Delta} B$ implies that $\kappa(B) < \kappa(\overline{B})$, we have $\omega \models B$. As ω was chosen arbitrarily, every world in Ω_{zero} is a model of B. Thus $\kappa'(B) = 0 < \kappa'(\overline{B})$, implying $\top \mid\!\sim_{\kappa'} B$. Because κ' was chosen arbitrarily among the extended c-representations of Δ', we have $\top \mid\!\sim^{ec}_{\Delta'} B$.

Since c-inference and extended c-inference coincide for strongly consistent belief bases, c-inference satisfies (wSM) as well. □

In summary, we showed that extended c-inference is the natural extension of c-inference to weakly consistent belief bases.

7. CSPs for extended c-representations

In this section, we investigate CSPs dealing with extended c-representations. In Section 7.1, we present a constraint system describing all extended c-representations of a belief base. Then we develop a simplification of this constraint system that takes the effects of conditionals in Δ^∞ into account right from the beginning. In Section 7.2, we show how extended c-inference can be realized by a CSP.

7.1. Describing extended c-representations by CSPs

The c-representations of a belief base Δ can conveniently be characterized by the solutions of a CSP (Kern-Isberner, 2001). In Beierle *et al*. (2018), the following modelling of c-representations as solutions of a CSP is introduced. For a belief base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ over Σ the CSP for c-representations of Δ, denoted by $CR_\Sigma(\Delta)$, on the constraint variables $\{\eta_1, \ldots, \eta_n\}$ ranging over $\mathbb{N}_0$ is given by the constraints cr_i^Δ, for all $i \in \{1, \ldots, n\}$:

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j \qquad (cr_i^\Delta)$$

The constraint (cr_i^Δ) is the constraint corresponding to the conditional $(B_i|A_i)$. The sum terms are induced by the worlds verifying and falsifying $(B_i|A_i)$, respectively. A solution of $CR_\Sigma(\Delta)$ is an n-tuple $(\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$. For a constraint satisfaction problem CSP, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR_\Sigma(\Delta))$, we denote the set of all solutions of $CR_\Sigma(\Delta)$. The solutions of $CR_\Sigma(\Delta)$ correspond to the c-representations of Δ.

Proposition 39 (soundness and completeness of $CR_\Sigma(\Delta)$, Beierle *et al.*, 2018). *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a belief base over Σ. Then we have:*

$$Mod^c_\Sigma(\Delta) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR_\Sigma(\Delta))\} \qquad (6)$$

If we want to construct a similar CSP for extended c-representations, we have to take worlds and formulas with infinite rank into account.

Definition 40 ($CR^{ex}_\Sigma(\Delta)$). *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a belief base over Σ. The CSP for extended c-representations of Δ, denoted by $CR^{ex}_\Sigma(\Delta)$, on the constraint variables $\{\eta_1, \ldots, \eta_n\}$ ranging over $\mathbb{N}_0 \cup \{\infty\}$ is given by the constraints $cr_i^{ex\,\Delta}$, for all $i \in \{1, \ldots, n\}$:*

$$\min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad or \quad \eta_i > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j \qquad (cr_i^{ex\,\Delta})$$

Again, each constraint ($cr_i^{ex\,\Delta}$) corresponds to the conditional $(B_i|A_i) \in \Delta$.

Proposition 41 (soundness and completeness of $CR^{ex}_\Sigma(\Delta)$). *Let* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ *be a weakly consistent belief base over* Σ. *Then we have*

$$Mod^{ec}_\Sigma(\Delta) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR^{ex}_\Sigma(\Delta))\} \tag{7}$$

Proof. **Soundness:** Let $\vec{\eta}$ be an impact vector in $Sol(CR^{ex}_\Sigma(\Delta))$. Because Δ is weakly consistent, there is a world ω that does not falsify any conditional in Δ; therefore, $\kappa_{\vec{\eta}}(\omega) = 0$ and $\kappa_{\vec{\eta}}$ is a ranking function. It is left to show that $\kappa_{\vec{\eta}}$ satisfies all conditionals in Δ.

Let $(B_i|A_i) \in \Delta$. There are three cases.

Case 1: $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$ and $\kappa_{\vec{\eta}}(A_iB_i) = \infty$

In this case $\kappa_{\vec{\eta}}(A_i) = \infty$ and therefore $\kappa_{\vec{\eta}} \models (B_i|A_i)$.

Case 2: $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$ and $\kappa_{\vec{\eta}}(A_iB_i) < \infty$

In this case, $\kappa_{\vec{\eta}}(A_i\overline{B_i} > \kappa_{\vec{\eta}}(A_iB_i) < \infty$ and therefore $\kappa_{\vec{\eta}} \models (B_i|A_i)$.

Case 3: $\kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$

In this case, $\min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$; hence, the condition in ($cr_i^{ex\,\Delta}$) before the *or* is not satisfied.

Because $\vec{\eta} \in Sol(CR^{ex}_\Sigma(\Delta))$ it must satisfy all constraints in $CR^{ex}_\Sigma(\Delta)$ including ($cr_i^{ex\,\Delta}$). Because the condition before the *or* is violated, it must hold that

$$\begin{aligned}
& \eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j \\
\Leftrightarrow \quad & \eta_i + \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j \\
\Leftrightarrow \quad & \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j \\
\Leftrightarrow \quad & \kappa_{\vec{\eta}}(A_i\overline{B_i}) > \kappa_{\vec{\eta}}(A_iB_i)
\end{aligned}$$

and therefore $\kappa_{\vec{\eta}} \models (B_i|A_i)$.

Completeness: Let $\kappa_{\vec{\eta}}$ be an extended c-representation of Δ with impact vector $\vec{\eta}$. We need to show that $\vec{\eta} \in Sol(CR^{ex}_\Sigma(\Delta))$, that is, that $\vec{\eta}$ satisfies every constraint ($cr_i^{ex\,\Delta}$) in $CR^{ex}_\Sigma(\Delta)$. Because $\kappa_{\vec{\eta}}$ is an extended c-representation of Δ, we have $\kappa_{\vec{\eta}} \models (B_i|A_i)$. This requires either (1) $\kappa_{\vec{\eta}}(A_i) = \infty$ or (2) $\kappa_{\vec{\eta}}(A_i\overline{B_i}) > \kappa_{\vec{\eta}}(A_iB_i)$. In case (1), it is $\min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$ and the condition before the *or* in ($cr_i^{ex\,\Delta}$) is satisfied. In case (2), we can see with the equivelence transformations in the *Soundness* part of this proof that the condition behind the *or* is satisfied. In both cases $\vec{\eta}$ satisfies ($cr_i^{ex\,\Delta}$). □

The requirement for weak consistency in Proposition 41 is necessary because for a belief base Δ that is not weakly consistent it holds that $CMod^{ec}_\Delta = \emptyset$ but $Sol(CR^{ex}_\Sigma(\Delta)) = \{(\infty, \ldots, \infty)\}$.

The resulting CSP $CR^{ex}_\Sigma(\Delta)$ is not a conjunction of inequalities any more, but it now contains disjunctions and is thus more complex. However, for the computation of extended c-inference, we can construct a simplified CSP $CRS^{ex}_\Sigma(\Delta)$ that still yields all extended c-representations relevant for c-inference. This is possible, because from Proposition 26 we already know which worlds must have rank infinity and which worlds may have finite rank in the extended c-representations of Δ. The simplified CSP not only uses fewer constraint variables but also fewer constraints than $CR^{ex}_\Sigma(\Delta)$ for weakly but not strongly consistent belief bases. Before stating $CRS^{ex}_\Sigma(\Delta)$, we show some proposition we will use for proving the correctness of $CRS^{ex}_\Sigma(\Delta)$.

We can assume the impacts of conditionals in Δ^∞ to be infinity.

Proposition 42. *Let Δ be a weakly consistent belief base with extended Z-partition $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$. Let $\vec{\eta}$ be impacts such that $\kappa_{\vec{\eta}}$ is an extended c-representation of Δ. Let $\vec{\eta}\,'$ be the impact vector defined by $\eta'_i = \infty$ if $(B_i|A_i) \in \Delta^\infty$ and $\eta'_i = \eta_i$ otherwise. Then $\kappa_{\vec{\eta}} = \kappa_{\vec{\eta}'}$.*

Proof. Let ω be a world. There are two cases.

Case 1: There is a conditional $(B_i|A_i) \in \Delta^\infty$ that is falsified by ω. Then $\kappa^z_\Delta(\omega) = \infty$ and therefore $\kappa_{\vec{\eta}}(\omega) = \infty$ by Proposition 26. Because $\eta'_i = \infty$ we have $\kappa_{\vec{\eta}'}(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j\overline{B}_j}} \eta'_j = \infty = \kappa_{\vec{\eta}}(\omega)$.

Case 2: There is no conditional in Δ^∞ that is falsified by ω. Because $\eta_i = \eta'_i$ for all i with $\omega \models A_j\overline{B}_j$ we have $\kappa_{\vec{\eta}'}(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j\overline{B}_j}} \eta'_j = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j\overline{B}_j}} \eta_j = \kappa_{\vec{\eta}}(\omega)$. □

Extended c-representations can assign rank infinity to conditionals without this being enforced by the belief base (cf. Proposition 23). In the following, we introduce conservative extended c-representations, which assign rank infinity only to worlds that also have rank infinity in the Z-ranking function κ^z_Δ.

Definition 43. *Let Δ be a belief base. Then $CMod^{ec}_\Delta$ is the set of conservative extended c-representations $\kappa_{\vec{\eta}}$ of Δ with $\kappa_{\vec{\eta}}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$.*

For c-inference, it is sufficient to take only conservative extended c-representations of a belief base into account.

Proposition 44. *Let Δ be a belief base. Then $A \mid\!\sim_\kappa B$ for all c-representations κ in $CMod^{ec}_\Delta$ iff $A \mid\!\sim_\kappa B$ holds for all c-representations κ in $Mod^{ec}_\Sigma(\Delta)$.*

Proof. Direction $\Leftarrow$: Observe that $CMod^{ec}_\Delta \subseteq Mod^{ec}_\Delta$. Therefore, if $A \mid\!\sim_\kappa B$ holds for all c-representations κ in Mod^{ec}_Δ, then $A \mid\!\sim_\kappa B$ holds for all c-representations κ in $CMod^{ec}_\Delta$.

Direction $\Rightarrow$: Show this by contraposition. Assume that $\kappa \in CMod^{ec}_\Delta$ with $A \not\mid\!\sim_\kappa B$. Using the construction of κ^f in the proof of Proposition 11 we can find a $\kappa' = \kappa^f$ that is a c-inference of Δ and satisfies $\mid\!\sim_{\kappa'} \subseteq \mid\!\sim_\kappa$. Therefore, if $A \not\mid\!\sim_\kappa B$ then $A \not\mid\!\sim_{\kappa'} B$. Hence, there is a c-representation κ' with $A \not\mid\!\sim_{\kappa'} B$. □

As indicated above, the c-representations in $CMod^{ec}_\Delta$ can then be characterized by a simplified CSP.

Definition 45 ($CRS^{ex}_\Sigma(\Delta)$). *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a belief base over Σ with the extended Z-partition $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$. Let*

$$J_\Delta = \{j \mid (B_j|A_j) \in \Delta \setminus \Delta^\infty \quad s.t.\ A_j\overline{B_j} \wedge \Big(\bigwedge_{(D|C) \in \Delta^\infty} (\overline{C} \vee D)\Big) \not\equiv \bot\}.$$

The simplified CSP for extended c-inference of Δ, denoted by $CRS^{ex}_\Sigma(\Delta)$, on the constraint variables $\{\eta_{j_1}, \ldots, \eta_{j_l}\}$, $j_k \in J_\Delta$ ranging over $\mathbb{N}_0$ is given by the constraints $crs^{ex\,\Delta}_j$, for all $j \in J_\Delta$:

$$\eta_i > \min_{\substack{\omega \in \Omega_\Sigma \\ \omega \models A_iB_i}} \sum_{\substack{j \in J_\Delta \\ j \neq i \\ \omega \models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega \in \Omega_\Sigma \\ \omega \models A_i\overline{B}_i}} \sum_{\substack{j \in J_\Delta \\ j \neq i \\ \omega \models A_j\overline{B_j}}} \eta_j \qquad (crs^{ex\,\Delta}_j)$$

The condition $A_j\overline{B_j} \wedge \left(\bigwedge_{(D|C) \in \Delta^\infty} (\overline{C} \vee D)\right) \not\equiv \bot$ in the definition of J_Δ is equivalent to there being an $\omega \in \Omega_{A_j\overline{B_j}}$ that does not falsify conditionals in Δ^∞.

Definition 46. *Let Δ be a belief base, $n = |\Delta|$, and let J_Δ be defined as above. For $\vec{\eta}^J \in Sol(CRS^{ex}_\Sigma(\Delta))$ let $\vec{\eta}^{J+\infty} \in (\mathbb{N}_0 \cup \{\infty\})^n$ be the impact vector with*

$$\eta_i^{J+\infty} = \begin{cases} \eta_i & \text{for } i \in J_\Delta \\ \infty & \text{otherwise.} \end{cases}$$

Then $Sol^{J+\infty}_\Delta := \{\vec{\eta}^{J+\infty} \mid \vec{\eta}^J \in Sol(CRS^{ex}_\Sigma(\Delta))\}$.

Proposition 47. (soundness and completeness of $CRS^{ex}_{\Sigma}(\Delta)$). *Let* Δ *be a weakly consistent belief base over* Σ. *Then*

$$CMod^{ec}_{\Sigma}(\Delta) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol^{J+\infty}_{\Delta}\}. \tag{8}$$

Proof. Let $EZP(\Delta) = (\Delta^0, \ldots, \Delta^k, \Delta^\infty)$, and let J_Δ be defined as in Definition 45.

Soundness: Let $\vec{\eta} \in Sol^{J+\infty}_{\Delta}$. By definition, there is a vector $\vec{\eta}^J \in Sol(CRS^{ex}_{\Sigma}(\Delta))$ such that $\vec{\eta} = \vec{\eta}^{J+\infty}$.

Because $\eta_i = \infty$ for every $(B_i|A_i) \in \Delta^\infty$ and due to Lemma 9, all worlds ω for which one of the conditionals in Δ^∞ is applicable have rank $\kappa_{\vec{\eta}}(\omega) = \infty$. Therefore, all conditionals in Δ^∞ are accepted by $\kappa_{\vec{\eta}}$.

For any conditional $(B_i|A_i) \in \Delta \setminus \Delta^\infty$, there is at least one world ω that verifies $(B_i|A_i)$ without falsifying a conditional in Δ^∞ (otherwise $(B_i|A_i)$ would not be tolerated by Δ^∞). Because every world that falsifies a conditional $(B_j|A_j)$ with $j \notin J_\Delta$ also falsifies a conditional in Δ^∞, the world ω does not falsify any such conditional $(B_j|A_j)$ with impact ∞. Therefore, $\kappa_{\vec{\eta}}(A_iB_i) < \infty$. If $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$ then $\kappa_{\vec{\eta}} \models (B_i|A_i)$. Otherwise, for $\kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$, there is a world that falsifies $(B|A)$ without falsifying a conditional in Δ^∞. In this case it is $i \in J_\Delta$ and the CSP $CRS^{ex}_{\Sigma}(\Delta)$ contains the constraint $(crs^{ex\,\Delta}_{j})$ which must hold for $\vec{\eta}^J$:

$$\begin{aligned}
& & \eta_i &> \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\in J_\Delta j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{j\in J_\Delta j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j \\
&\Leftrightarrow & \eta_i + \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{j\in J_\Delta j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j &> \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\in J_\Delta j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j \\
&\Leftrightarrow & \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j &> \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j \\
&\overset{(*)}{\Leftrightarrow} & \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B_i}}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j &> \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j \\
&\Leftrightarrow & \kappa_{\vec{\eta}}(A_i\overline{B_i}) &> \kappa_{\vec{\eta}}(A_iB_i).
\end{aligned}$$

Therefore, $\kappa_{\vec{\eta}} \models (B_i|A_i)$.

The equivalence $(\,*\,)$ holds, because there is a model for $A_i\dot{B}_i$ that does not falsify a conditional in Δ^∞, we have $\eta_j = \infty$ for all $(B_j|A_j)$ with $j \notin J_\Delta$, and therefore

$$\min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\dot{B}_i}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j = \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\dot{B}_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j.$$

For any world ω with $\kappa^z_\Delta(\omega) < \infty$ it holds that all conditionals in Δ^∞ are not applicable in ω. Therefore, $\kappa_{\vec{\eta}}(\omega)$ is the sum of some of the impacts in $\vec{\eta}^J$; and because $\vec{\eta}^J \in \mathbb{N}^l_0$ we have $\kappa_{\vec{\eta}}(\omega) < \infty$.

In summary, $\kappa_{\vec{\eta}} \in CMod^{ec}_{\Delta}$.

Completeness: Let $\kappa \in CMod^{ec}_{\Delta}$ be an extended c-representation. Let $\vec{\eta} \in (\mathbb{N}_0 \cup \infty)^n$ be an impact vector such that $\kappa = \kappa_{\vec{\eta}}$. Because of Proposition 42, w.l.o.g. we can assume $\eta_i = \infty$ for all $(B_i|A_i) \in \Delta^\infty$. Furthermore, w.l.o.g, we can assume $\eta_i = \infty$ for all conditionals $(B_i|A_i) \in \Delta \setminus \Delta^\infty$ which are falsified only by worlds ω that also falsify a conditional in Δ^∞ – all worlds for which these impacts apply already have rank ∞ because of the impacts for Δ^∞.

The vector $\vec{\eta}$ is a combination of a vector $\vec{\eta}^J$ of impacts η_j for $j \in J_\Delta$, and a vector $(\infty, \ldots, \infty)$ of size $n - |J_\Delta|$ of impacts for conditionals $(B_j|A_j)$ with $j \notin J_\Delta$.

For every $i \in J_\Delta$, by construction of J_Δ there is at least one world ω falsifying $(B_i|A_i)$ without falsifying a conditional in Δ^∞. Then, $\kappa^z_\Delta(\omega) < \infty$ because ω falsifies no conditionals in Δ^∞ and due to Lemma 15; therefore $\eta_i < \kappa_{\vec{\eta}}(\omega) < \infty$ because $\kappa_{\vec{\eta}} \in CMod^{ec}_{\Delta}$. Hence, $\vec{\eta}^J \in \mathbb{N}_0$.

It is left to show that $\vec{\eta}^J$ is a solution of $CRS^{ex}_{\Sigma}(\Delta)$, that is, that for every $j \in J_\Delta$ it satisfies the constraint $(crs^{ex\,\Delta}_{j})$. As $\kappa_{\vec{\eta}}$ is a model of Δ, it satisfies the conditional $(B_j|A_j) \in \Delta$. By construction of J_Δ, there is at least one world ω falsifying $(B_j|A_j)$ without falsifying a conditional in Δ^∞. As established above, the rank of such a world in $\kappa_{\vec{\eta}}$ is finite, and thus $\kappa_{\vec{\eta}}(A)$ is finite. To satisfy$(B_j|A_j)$ it is necessary that $\kappa_{\vec{\eta}}(A_i\overline{B_i}) > \kappa_{\vec{\eta}}(A_iB_i)$. Using the equivalence transformation in the *Soundness* part of this proof, we obtain that $(crs_j^{ex\,\Delta})$ holds for η_j. ☐

Propositions 44 and 47 imply the following result.

Proposition 48. *Let Δ be a weakly consistent belief base. Then $A \mid\!\sim^{ec}_{\Delta} B$ iff $A \mid\!\sim_{\kappa_{\vec{\eta}}} B$ for every $\vec{\eta} \in Sol_{\Delta}^{J+\infty}$.*

The following example illustrates how $CRS_{\Sigma}^{ex}(\Delta)$ is simpler than $CR_{\Sigma}^{ex}(\Delta)$.

Example 49. *Let $\Sigma = \{a, b, c\}$ and $\Delta = \{(\bot|a), (\overline{a}|b), (b|c)\}$. The CSP $CR_{\Sigma}^{ex}(\Delta)$ over $\eta_1, \eta_1, \eta_3 \in \mathbb{N}_0 \cup \{\infty\}$ contains the constraints*

$$(cr_1^{ex\,\Delta}) \quad \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models a}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad or \quad \eta_1 > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models a\wedge\bot}} \sum_{\substack{j\neq 1 \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models a\wedge\top}} \sum_{\substack{j\neq 1 \\ \omega\models A_j\overline{B_j}}} \eta_j,$$

$$(cr_2^{ex\,\Delta}) \quad \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models b}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad or \quad \eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models b\overline{a}}} \sum_{\substack{j\neq 2 \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models ba}} \sum_{\substack{j\neq 2 \\ \omega\models A_j\overline{B_j}}} \eta_j,$$

$$(cr_3^{ex\,\Delta}) \quad \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models c}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad or \quad \eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models cb}} \sum_{\substack{j\neq 3 \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models c\overline{b}}} \sum_{\substack{j\neq 3 \\ \omega\models A_j\overline{B_j}}} \eta_j.$$

The extended Z-partition of Δ is $EZP(\Delta) = (\Delta^0, \Delta^\infty)$ with $\Delta^0 = \{(\overline{a}|b), (b|c)\}$ and $\Delta^\infty = \{(\bot|a)\}$. The conditional $(\overline{a}|b)$ cannot be falsified without also falsifying $(\bot|a) \in \Delta^\infty$. Therefore, $J_\Delta = \{3\}$ and the CSP $CRS_{\Sigma}^{ex}(\Delta)$ over $\eta_3 \in \mathbb{N}_0$ contains only the constraint

$$(crs_3^{ex\,\Delta}) \qquad \eta_3 > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models bc}} \sum_{\substack{j\in J_\Delta \\ j\neq 3 \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models b\overline{c}}} \sum_{\substack{j\in J_\Delta \\ j\neq 3 \\ \omega\models A_j\overline{B_j}}} \eta_j$$

which simplifies to $\eta_3 > 0$. For $\vec{\eta} \in Sol_{\Delta}^{J+\infty}$ it holds that $\eta_1 = \eta_2 = \infty$ and $\eta_3 \in Sol(CRS_{\Sigma}^{ex}(\Delta))$.

7.2. Characterizing extended c-inference by a CSP

In Beierle *et al*. (2018) a method is developed that realizes c-inference as a CSP. The idea of this approach is that in order to check whether $A \mid\!\sim^{c}_{\Delta} B$ holds, a constraint encoding that $A \mid\!\sim_{\kappa_{\vec{\eta}}} B$ does not hold is added to $CR_\Sigma(\Delta)$. If the resulting CSP is unsovable, $A \mid\!\sim_{\kappa_{\vec{\eta}}} B$ holds for all solutions $\vec{\eta}$ of $CR_\Sigma(\Delta)$. Based on this idea, we develop a CSP that allows doing something similar for extended c-inference.

First we need a constraint encoding that $A \mid\!\sim_{\kappa_{\vec{\eta}}} B$ does not hold for an extended c-representation $\kappa_{\vec{\eta}}$.

Definition 50. *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a belief base and let J_Δ be as defined in Definition 45. The constraint $\neg CR_\Delta(B|A)$ is given by*

$$\min_{\omega\models AB} \sum_{\substack{i\in J_\Delta \\ \omega\models A_i\overline{B_i}}} \eta_i \geqslant \min_{\omega\models A\overline{B}} \sum_{\substack{i\in J_\Delta \\ \omega\models A_i\overline{B_i}}} \eta_i. \tag{9}$$

Using this constraint and the CSP $CRS_{\Sigma}^{ex}(\Delta)$ developed in Section 7.1 we can check if $A \mid\!\sim^{ec}_{\Delta} B$ with the following proposition.

Proposition 51. *Let Δ be weakly consistent. Then $A \mid\!\sim^{ec}_{\Delta} B$ iff (i) $\kappa^z_\Delta(A\overline{B}) = \infty$ or (ii) $\kappa^z_\Delta(A\overline{B}) < \infty$, $\kappa^z_\Delta(AB) < \infty$ and $CRS_{\Sigma}^{ex}(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable.*

Proof. **Direction** $\Rightarrow$: Assume that $A \mid\!\sim^{ec}_{\Delta} B$ and that $\kappa^z_\Delta(A\overline{B}) < \infty$. Then $\kappa(A\overline{B}) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$ by the definition of $CMod^{ec}_\Delta$. Therefore, $\kappa(A) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$. Furthermore, $A \mid\!\sim^{ec}_{\Delta} B$ implies that for every $\kappa \in CMod^{ec}_\Delta$, we have $A \mid\!\sim_\kappa B$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_\Delta$, and because of

Proposition 47 $\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$ for every $\vec{\eta} \in Sol_{\Delta}^{J+\infty}$. We have

$$\begin{aligned} & \kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B}) \\ \Leftrightarrow \quad & \min_{\omega \models AB} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i < \min_{\omega \models A\overline{B}} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \\ \overset{(*)}{\Leftrightarrow} \quad & \min_{\omega \models AB} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i < \min_{\omega \models A\overline{B}} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i . \end{aligned}$$

Equivalence $(*)$ holds because the ranks of the minimal models of $A\ B$ and $A\overline{B}$ are finite and therefore do not violate a conditional $(B_i|A_i)$ with $i \notin J_\Delta$.

Therefore, $\neg CR_\Delta(B|A)$ does not hold for any solution of $CRS_\Sigma^{ex}(\Delta)$, implying that $CRS_\Sigma^{ex}(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable.

Direction $\Leftarrow$: Assume that either $\kappa_\Delta^z(A\overline{B}) = \infty$ or $\big(\kappa_\Delta^z(AB) < \infty$ and $CRS_\Sigma^{ex}(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable $\big)$. There are three cases.

Case 1: $\kappa_\Delta^z(AB) = \infty$ and $\kappa_\Delta^z(A\overline{B}) = \infty$

Then $\kappa_\Delta^z(A) = \infty$ and, by Proposition 26, $\kappa(A) = \infty$ for every $\kappa \in Mod_\Delta^{ec}$. Therefore $A \mid\!\sim_\Delta^{ec} B$.

Case 2: $\kappa_\Delta^z(AB) < \infty$ and $\kappa_\Delta^z(A\overline{B}) = \infty$

Then, by the definition of $CMod_\Delta^{ec}$, we have $\kappa(AB) < \infty$ and, by Proposition 26, $\kappa(A\overline{B}) = \infty$ for every $\kappa \in CMod_\Delta^{ec}$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod_\Delta^{ec}$ and hence $A \mid\!\sim_\Delta^{ec} B$ by Proposition 44.

Case 3: $\kappa_\Delta^z(A\overline{B}) < \infty$

Then, by assumption, $CRS_\Sigma^{ex}(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable and $\kappa_\Delta^z(AB) < \infty$. This implies that $\neg CR_\Delta(B|A)$ is false for every $\vec{\eta}^J \in Sol(CRS_\Sigma^{ex}(\Delta))$. In this case, using the equivalence transformations in the part of the proof for *Direction* $\Rightarrow$, we have $\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$ for every $\vec{\eta} \in Sol_\Delta^{J+\infty}$. With Proposition 48 it follows that $A \mid\!\sim_\Delta^{ec} B$. □

This realization of extended c-inference by a CSP yields a starting point for an implementation of extended c-inference as a SAT or SMT problem (Beierle *et al.*, 2022; von Berg *et al.*, 2023).

8. Syntax splitting

The concept of *syntax splittings* was originally developed by Parikh (1999) describing that a belief set contains independent information over different parts of the signature. The notion of syntax splitting was later extended to other representations of beliefs (Kern-Isberner & Brewka, 2017; Kern-Isberner *et al.*, 2020). In Kern-Isberner *et al.* (2020) not only inductive inference operators were introduced, but also the postulates (Rel), (Ind), (SynSplit) for inductive inference operators that govern inference from strongly consistent belief bases with syntax splitting. Notably, c-inference satisfies these postulates. In this section we show that also extended c-inference respects syntax splittings on belief bases.

Definition 52 (syntax splitting for belief bases, Kern-Isberner *et al.*, 2020). *Let Δ be a belief base over Σ. A partition $\{\Sigma_1, \ldots, \Sigma_n\}$ of Σ is a syntax splitting for Δ if there is a partition $\{\Delta_1, \ldots, \Delta_n\}$ of Δ s.t. $\Delta_i \subseteq (\mathcal{L}|\mathcal{L})_{\Sigma_i}$ for every $i = 1, \ldots, n$.*

In this paper, we only consider syntax splittings with two parts. Such a splitting $\{\Sigma_1, \Sigma_2\}$ of Δ with corresponding partition $\{\Delta_1, \Delta_2\}$ is denoted as (Kern-Isberner *et al.*, 2020) $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$. Results for syntax splittings with more than two parts can be obtained by iteratively applying the postulates presented here.

Here, we present slightly adapted versions of the splitting postulates, (Rel^+), (Ind^+), and (SynSplit^+) (Kern-Isberner *et al.*, 2020; Haldimann *et al.*, 2023), that are intended for inference operators that are defined for all belief bases, including only weakly consistent belief bases.

For belief bases with syntax splitting, the postulate relevance (Rel$^+$) requires that for an inference using only atoms from one part of the syntax splitting only conditionals from the corresponding part of the belief base are relevant.

(Rel$^+$) An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies (Rel$^+$) if for a weakly consistent $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, for $i = 1, 2$, and for $A, B \in \mathcal{L}_{\Sigma_i}$ we have

$$A \mid\!\sim_\Delta B \quad \text{iff} \quad A \mid\!\sim_{\Delta_i} B.$$

The postulate independence (Ind$^+$) requires that an inference using only atoms from one part of the syntax splitting should be drawn independently of additional beliefs about other parts of the splitting.

(Ind$^+$) An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies (Ind$^+$) if for any weakly consistent $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, and for $i, j \in \{1, 2\}, i \neq j$ and for any $A, B \in \mathcal{L}_{\Sigma_i}$, $D \in \mathcal{L}_{\Sigma_j}$ such that $D \not\mid\!\sim_\Delta \bot$, we have

$$A \mid\!\sim_\Delta B \quad \text{iff} \quad AD \mid\!\sim_\Delta B.$$

The difference of these postulates to (Rel) and (Ind) is that also weakly consistent belief bases are considered. Additionally, in (Ind$^+$) the requirement $D \not\mid\!\sim_\Delta \bot$ was added. Otherwise, the postulate would have some clearly unintended implications. For any formula $A \in \mathcal{L}_{\Sigma_1}$ with (Ind) we would have $AD \mid\!\sim_\Delta \bot$ for any $D \in \mathcal{L}_{\Sigma_2}$ with $D \mid\!\sim_\Delta \bot$. Then, (Ind) would imply that $A \not\mid\!\sim_\Delta \bot$. The condition $D \not\mid\!\sim_\Delta \bot$ avoids this unintended consequence.

(SynSplit$^+$) An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies (SynSplit$^+$) if it satisfies both (Rel$^+$) and (Ind$^+$).

For our proof that extended c-inference complies with syntax splitting, we need some lemmas on the behaviour of c-representations in the context of a syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$. Differing from the proof that c-inference satisfies (SynSplit) in Kern-Isberner *et al*. (2020), in the following we argue about the sets of extended c-inferences $Mod^{ec}_\Sigma(\Delta)$ directly instead of the sets of solutions of the corresponding CSPs. This is mainly because the CSPs characterizing all extended c-representations are more involved than the CSPs for c-representations. First observe that the combination of any extended c-inference of Δ_1 with an extended c-inference of Δ_2 is an extended c-inference of Δ.

Lemma 53. *Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be a weakly consistent belief base with syntax splitting. Let $\kappa_1 \in Mod^{ec}_{\Sigma_1}(\Delta_1)$ and $\kappa_2 \in Mod^{ec}_{\Sigma_2}(\Delta_2)$. Then $\kappa_\oplus = \kappa_1 \oplus \kappa_2$ is an extended c-representation of Δ, that is, $\kappa_\oplus \in Mod^{ec}_\Sigma(\Delta)$.*

Proof. Let $(B_1|A_1), \ldots, (B_k|A_k)$ be the conditionals in Δ_1 and let $(B_{k+1}|A_{k+1}), \ldots, (B_n|A_n)$ be the conditionals in Δ_2. Let $\vec{\eta}^1$ be an impact vector inducing κ_1 and let $\vec{\eta}^2$ be an impact vector inducing κ_2. Let ω be any world in Ω_Σ. Because of the syntax splitting, $\omega_{|\Sigma_1}$ falsifies the same worlds in Δ_1 as ω and $\omega_{|\Sigma_2}$ falsifies the same worlds in Δ_2 as ω. Let $\vec{\eta}$ be the impact vector for Δ that combines the impacts from $\vec{\eta}^1$ and $\vec{\eta}^2$. We have that

$$\begin{aligned} \kappa_\oplus &= \kappa_1(\omega_{|\Sigma_1}) + \kappa_2(\omega_{|\Sigma_2}) \\ &= \sum_{\substack{1 \leqslant i \leqslant k \\ \omega_{|\Sigma_1} \models A_i\overline{B_i}}} \eta_i + \sum_{\substack{k+1 \leqslant i \leqslant n \\ \omega_{|\Sigma_2} \models A_i\overline{B_i}}} \eta_i \\ &= \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \eta_i = \kappa_{\vec{\eta}}. \end{aligned} \tag{10}$$

Additionally, because $\kappa_1 = \kappa_{\oplus|\Sigma_1}$ and $\kappa_2 = \kappa_{\oplus|\Sigma_2}$ (see Lemma 4), Lemma 2 yields that $\kappa_\oplus$ models the conditionals in Δ_1 and in Δ_2. Therefore, $\kappa_\oplus$ is an extended c-representation of Δ. □

In the other direction, Lemma 54 shows how an extended c-representation of Δ splits for certain formulas.

Lemma 54. *Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be a weakly consistent belief base with syntax splitting. Let $\kappa_{\vec{\eta}} \in Mod^{ec}_{\Sigma}(\Delta)$ be an extended c-representation induced by impact vector $\vec{\eta}$. Let $\vec{\eta}^1$ be the impact vector containing the impacts from $\vec{\eta}$ for Δ_1 and let $\vec{\eta}^2$ be the impact vector containing the impacts from $\vec{\eta}$ for Δ_2. Then, for $X \in \mathcal{L}_{\Sigma_1}$, $Y \in \mathcal{L}_{\Sigma_2}$ we have $\kappa(XY) = \kappa_{\vec{\eta}^1}(X) + \kappa_{\vec{\eta}^2}(Y)$.*

Proof. Let $(B_1|A_1), \ldots, (B_k|A_k)$ be the conditionals in Δ_1 and let $(B_{k+1}|A_{k+1}), \ldots, (B_n|A_n)$ be the conditionals in Δ_2. For any $\omega \in \Omega_\Sigma$ we have that

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \eta_i = \sum_{\substack{1 \leqslant i \leqslant k \\ \omega \models A_i\overline{B_i}}} \eta_i + \sum_{\substack{k+1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \eta_i = \kappa_{\vec{\eta}^1}(\omega) + \kappa_{\vec{\eta}^2}(\omega).$$

Because $X \in \mathcal{L}_{\Sigma_1}$ and $Y \in \mathcal{L}_{\Sigma_2}$ we have that $Mod_{\,\Sigma}(XY) = \{(\omega^x \cdot \omega^y) \mid \omega^x \in Mod_{\,\Sigma_1}(X), \omega^y \in Mod_{\,\Sigma_2}(Y)\}$. In summary, we have that

$$\begin{aligned}
\kappa(XY) &= \min\{\kappa(\omega) \mid \omega \in Mod_{\,\Sigma}(XY)\} \\
&= \min\{\kappa(\omega^x \cdot \omega^y) \mid \omega^x \in Mod_{\,\Sigma_1}(X), \omega^y \in Mod_{\,\Sigma_2}(Y)\} \\
&= \min\{\kappa_{\vec{\eta}^1}(\omega^x) + \kappa_{\vec{\eta}^2}(\omega^y) \mid \omega^x \in Mod_{\,\Sigma_1}(X), \omega^y \in Mod_{\,\Sigma_2}(Y)\} \\
&= \min\{\kappa_{\vec{\eta}^1}(\omega^x) \mid \omega^x \in Mod_{\,\Sigma_1}(X)\} + \min\{\kappa_{\vec{\eta}^2}(\omega^y) \mid \omega^y \in Mod_{\,\Sigma_2}(Y)\} \\
&= \kappa_{\vec{\eta}^1}(X) + \kappa_{\vec{\eta}^2}(Y).
\end{aligned}$$

□

Lemma 55 states that marginalizing an extended c-representation of Δ to the subsignature Σ_i, $i \in \{1, 2\}$ leads to an extended c-representation of Δ_i.

Lemma 55. *Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be a weakly consistent belief base with syntax splitting. If $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ then $\kappa_{|\Sigma_1} \in Mod^{ec}_{\Sigma_1}(\Delta_1)$.*

Proof. Let $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$. By Lemma 2, $\kappa_{|\Sigma_1}$ models the same conditionals in $(\mathcal{L}|\mathcal{L})_{\Sigma_1}$ as κ. Especially, $\kappa_{|\Sigma_1}$ models all conditionals in Δ_1. It remains to be shown that $\kappa_{|\Sigma_1}$ can be constructed from integer impacts.

For this let $(B_1|A_1), \ldots, (B_k|A_k)$ be the conditionals in Δ_1 and let $(B_{k+1}|A_{k+1}), \ldots, (B_n|A_n)$ be the conditionals in Δ_2. Let $\vec{\eta}$ be an impact vector inducing κ, and let $\vec{\eta}^1$ be the impact vector containing the impacts from $\vec{\eta}$ for Δ_1. Let $\omega^0 \in \Omega_\Sigma$ be a world that does not falsify any conditional in Δ (see Lemma 9). Let ω' be any world in Ω_{Σ_1}. All worlds ω with $\omega_{|\Sigma_1} = \omega'$ falsify the same conditionals in Δ_1. The world $\omega^* = (\omega' \cdot {\omega^0}_{|\Sigma_2})$ falsifies no conditional in Δ_2 and is thus one of the worlds with the lowest rank in κ that coincides with ω' on Σ_1. We have that

$$\begin{aligned}
\kappa_{|\Sigma_1}(\omega') &= \min\{\kappa(\omega) \mid \omega_{|\Sigma_1} = \omega'\} \\
&= \min\{\sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \eta_i \mid \omega_{|\Sigma_1} = \omega'\} \\
&= \sum_{\substack{1 \leqslant i \leqslant n \\ \omega^* \models A_i\overline{B_i}}} \eta_i = \sum_{\substack{1 \leqslant i \leqslant k \\ \omega' \models A_i\overline{B_i}}} \eta_i = \kappa_{\vec{\eta}^1}(\omega').
\end{aligned}$$

Therefore, $\kappa_{|\Sigma_1}$ is an extended c-representation of Δ_1. □

One of the key observations for proving that c-inference satisfies (SynSplit) in Kern-Isberner *et al.* (2020) is Kern-Isberner *et al.* (2020), Proposition 8. Given a belief base with syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, this proposition states that the c-representations of Δ are exactly the combinations of the c-representations of Δ_1 with the c-representations of Δ_2. In combination, the Lemmas 53, 54, and 55 yield a corresponding observation for extended c-representations as expressed in the following lemma.

Lemma 56. *Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be a weakly consistent belief base with syntax splitting. Then $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ iff there are $\kappa_1 \in Mod^{ec}_{\Sigma_1}(\Delta_1)$ and $\kappa_2 \in Mod^{ec}_{\Sigma_2}(\Delta_2)$ such that $\kappa = \kappa_1 \oplus \kappa_2$.*

Proof. Direction $\Leftarrow$ follows from Lemma 54. In the other direction $\Rightarrow$, with Lemma 54 we can see that there are $\kappa_1 : \Omega_{\Sigma_1} \to \mathbb{N} \cup \{\infty\}$ and $\kappa_1 : \Omega_{\Sigma_1} \to \mathbb{N} \cup \{\infty\}$ such that $\kappa = \kappa_1 \oplus \kappa_2$. By Lemma 4 we have that $\kappa_i = (\kappa_1 \oplus \kappa_2)_{|\Sigma_i} = \kappa_{|\Sigma_i}$ for $i \in \{1,2\}$ and therefore, with Lemma 55, that $\kappa_1 \in Mod^{ec}_{\Sigma_1}(\Delta_1)$ and $\kappa_2 \in Mod^{ec}_{\Sigma_1}(\Delta_1)$. □

Extended c-inference does not change if unused atoms are added to or removed from the signature. This is captured by the following Lemma 57.

Lemma 57. *Let Σ be a signature, $\Sigma' \subseteq \Sigma$, and Δ be a belief base over Σ'. For $A, B \in \mathcal{L}_{\Sigma'}$ we have $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ iff $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ'.*

Proof. Show both directions of the *iff*.

Direction $\Rightarrow$: Let $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ. Let $\kappa' \in Mod^{ec}_{\Sigma'}(\Delta)$. We can see Δ as a belief base with syntax splitting $\Delta = \Delta \bigcup_{\Sigma',\Sigma\setminus\Sigma'} \emptyset$. The only extended c-representation in $Mod^{ec}_{\Sigma\setminus\Sigma'}(\emptyset)$ is κ^0 with $\kappa^0(\omega) = 0$ for all $\omega \in \Omega_{\Sigma\setminus\Sigma'}$. By Lemma 53, $\kappa_{\oplus} = \kappa' \oplus \kappa^0$ is in $Mod^{ec}_{\Sigma}(\Delta)$, and therefore $A \mid\!\sim_{\kappa_{\oplus}} B$. By Lemma 5, this implies that $A \mid\!\sim_{\kappa'} B$ because $\kappa' = \kappa_{\oplus|\Sigma'}$ (see Lemma 4). Hence, in summary we have that $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ'.

Direction $\Leftarrow$: Let $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ'. Let $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ be an extended c-representation. Again, we can see Δ as a belief base with syntax splitting $\Delta = \Delta \bigcup_{\Sigma',\Sigma\setminus\Sigma'} \emptyset$. By Lemma 55, $\kappa' = \kappa_{|\Sigma'}$ is an extended c-representation in $Mod^{ec}_{\Sigma'}(\Delta)$, and therefore $A \mid\!\sim_{\kappa'} B$. Then Lemma 5 implies that $A \mid\!\sim_{\kappa} B$. Hence, in summary we have that $A \mid\!\sim^{ec}_{\Delta} B$ with respect to Σ. □

Using the lemmas above, we can now show that extended c-inference satisfies (Rel$^+$). Observe that the proof for the next Proposition 58 does not need to deal explicitly with the case that a formula/world is infeasible (i.e., has rank ∞). In this proof, all situations where a world or formula has rank ∞ are already covered by the underlying definitions and results.

Proposition 58. *Extended c-inference satisfies (Rel$^+$).*

Proof. Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be weakly consistent. W.l.o.g. let $i = 1$ and $A, B \in \mathcal{L}_{\Sigma_1}$. We need to show that $A \mid\!\sim^{ec}_{\Delta} B$ iff $A \mid\!\sim^{ec}_{\Delta_1} B$.

Direction $\Rightarrow$: Assume that $A \mid\!\sim^{ec}_{\Delta} B$. Because of Lemma 57 it is sufficient to show that $A \mid\!\sim^{ec}_{\Delta_1} B$ w.r.t. Σ_1. Let $\kappa_1 \in Mod^{ec}_{\Sigma_1}(\Delta_1)$ be any extended c-representation of Δ_1. We need to show that $A \mid\!\sim_{\kappa_1} B$.

Let $\kappa_2 \in Mod^{ec}_{\Sigma_2}(\Delta_2)$ and $\kappa_{\oplus} = \kappa_1 \oplus \kappa_2$. By Lemma 53 we have $\kappa_{\oplus} \in Mod^{ec}_{\Sigma}(\Delta)$. Therefore, by assumption we have $A \mid\!\sim_{\kappa_{\oplus}} B$. Because $\kappa_1 = \kappa_{\oplus|\Sigma_1}$, with Lemma 5 we have that $A \mid\!\sim_{\kappa_1} B$.

Direction $\Leftarrow$: Assume that $A \mid\!\sim^{ec}_{\Delta_1} B$ (w.r.t. Σ). Let $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ be any extended c-representation of Δ. We need to show that $A \mid\!\sim_{\kappa} B$.

With Lemma 55 we have that $\kappa_1 = \kappa_{|\Sigma_1}$ is an extended c-representation of Δ_1. With Lemma 57 we have that $A \mid\!\sim^{ec}_{\Delta_1} B$ w.r.t. Σ_1, and therefore $A \mid\!\sim_{\kappa_1} B$. Using Lemma 5 we have that $A \mid\!\sim_{\kappa} B$. □

For proving (Ind$^+$), in contrast to proving (Rel$^+$) in Proposition 58, we have to distinguish explicitly between the case that an entailment holds because its antecedent is infeasible and the case that an entailment holds because its verification has a lower rank than its falsification.

Proposition 59. *Extended c-inference satisfies (Ind$^+$).*

Proof. Let $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ be weakly consistent. W.l.o.g. let $i = 1, j = 2$ and $A, B \in \mathcal{L}_{\Sigma_1}, D \in \mathcal{L}_{\Sigma_2}$ such that $D \not\mid\!\sim^{ec}_{\Delta} \bot$. We need to show that $A \mid\!\sim^{ec}_{\Delta} B$ iff $AD \mid\!\sim^{ec}_{\Delta} B$.

Direction $\Rightarrow$: Assume $A \mid\!\sim^{ec}_{\Delta} B$. We show $AD \mid\!\sim_{\kappa} B$ for every $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$.

Let $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ be any extended c-representation of Δ. Because $A \mathrel{|\!\sim}^{ec}_{\Delta} B$ we have that $A \mathrel{|\!\sim}_{\kappa} B$. Let $\vec{\eta}$ be the impact vector inducing κ, that is, $\kappa = \kappa_{\vec{\eta}}$. Because $\Delta = \Delta_1 \cup \Delta_2$, we can sort the impacts in $\vec{\eta}$ into an impact vector $\vec{\eta}^1$ for Δ_1 and an impact vector $\vec{\eta}^2$ for Δ_2. We can distinguish three cases.

Case 1: $\kappa_{\vec{\eta}}(A) = \infty$ Applying Lemma 54, we have $\kappa_{\vec{\eta}}(A) = \kappa_{\vec{\eta}}(A \wedge \top) = \kappa_{\vec{\eta}^1}(A) + \kappa_{\vec{\eta}^2}(\top)$. Because Δ is weakly consistent, there is at least one world ω that does not falsify a conditional in Δ. Therefore, we have $\kappa_{\vec{\eta}^2}(\top) = 0$. Hence, $\kappa_{\vec{\eta}^1}(A) = \infty$. This implies $\kappa_{\vec{\eta}}(AD) = \kappa_{\vec{\eta}^1}(A) + \kappa_{\vec{\eta}^2}(D) = \infty + \kappa_{\vec{\eta}^2}(D) = \infty$. Thus, $AD \mathrel{|\!\sim}_{\kappa} B$.

Case 2: $\kappa_{\vec{\eta}}(A) < \infty$ and $\kappa_{\vec{\eta}}(D) = \infty$ With a similar argumentation as in Case 1, we have $\kappa_{\vec{\eta}^2}(D) = \infty$. This implies $\kappa_{\vec{\eta}}(AD) = \kappa_{\vec{\eta}^1}(A) + \kappa_{\vec{\eta}^2}(D) = \kappa_{\vec{\eta}^1}(A) + \infty = \infty$. Thus, $AD \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}} B$.

Case 3: $\kappa_{\vec{\eta}}(A) < \infty$ and $\kappa_{\vec{\eta}}(D) < \infty$ Because $A \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}} B$ in this case it is necessary that $\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$. Applying Lemma 54, we have $\kappa_{\vec{\eta}}(AB) = \kappa_{\vec{\eta}}(AB \wedge \top) = \kappa_{\vec{\eta}^1}(AB) + \kappa_{\vec{\eta}^2}(\top)$. Because Δ is weakly consistent, with Lemma 9 there is at least one world ω that does not falsify a conditional in Δ. Therefore, we have $\kappa_{\vec{\eta}^2}(\top) = 0$. Hence, $\kappa_{\vec{\eta}^1}(AB) = \kappa_{\vec{\eta}}(AB)$. Analogously, $\kappa_{\vec{\eta}^1}(A\overline{B}) = \kappa_{\vec{\eta}}(A\overline{B})$ and $\kappa_{\vec{\eta}^2}(D) = \kappa_{\vec{\eta}}(D) < \infty$. Therefore, $\kappa_{\vec{\eta}^1}(AB) < \kappa_{\vec{\eta}^1}(A\overline{B})$. Lemma 54 also yields $\kappa_{\vec{\eta}}(ABD) = \kappa_{\vec{\eta}^1}(AB) + \kappa_{\vec{\eta}^2}(D)$ and $\kappa_{\vec{\eta}}(A\overline{B}D) = \kappa_{\vec{\eta}^1}(A\overline{B}) + \kappa_{\vec{\eta}^2}(D)$. Together, we have that $\kappa_{\vec{\eta}}(ABD) = \kappa_{\vec{\eta}^1}(AB) + \kappa_{\vec{\eta}^2}(D) < \kappa_{\vec{\eta}^1}(A\overline{B}) + \kappa_{\vec{\eta}^2}(D) = \kappa_{\vec{\eta}}(A\overline{B}D)$ and thus $AD \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}} B$.

Direction $\Leftarrow$: Assume $AD \mathrel{|\!\sim}^{ec}_{\Delta} B$. We show $A \mathrel{|\!\sim}_{\kappa} B$ for every $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$.

Let $\kappa \in Mod^{ec}_{\Sigma}(\Delta)$ be any extended c-representation of Δ. Because $AD \not\mathrel{|\!\sim}^{ec}_{\Delta} B$ we have $AD \mathrel{|\!\sim}_{\kappa} B$. Because $D \not\mathrel{|\!\sim}^{ec}_{\Delta} \bot$, there is a $\kappa' \in Mod^{ec}_{\Sigma}(\Delta)$ such that $D \not\mathrel{|\!\sim}_{\kappa'} \bot$ which implies $\kappa'(D) < \infty$. Using Lemma 55 we have $\kappa'_{|\Sigma_2} \in Mod^{ec}_{\Sigma_2}(\Delta_2)$. With Lemma 1 we have $\kappa'_{|\Sigma_2}(D) = \kappa'(D)$. Analogously, we have $\kappa_{|\Sigma_1} \in Mod^{ec}_{\Sigma_1}(\Delta_1)$ and $\kappa_{|\Sigma_1}(A) = \kappa(A)$ and $\kappa_{|\Sigma_1}(AB) = \kappa(AB)$ and $\kappa_{|\Sigma_1}(A\overline{B}) = \kappa(A\overline{B})$. Let $\kappa_{\oplus} = \kappa_{|\Sigma_1} \oplus \kappa'_{|\Sigma_2}$. With Lemma 53 we have $\kappa_{\oplus} \in Mod^{ec}_{\Sigma}(\Delta)$. Because $AD \mathrel{|\!\sim}^{ec}_{\Delta} B$ this entails that $AD \mathrel{|\!\sim}_{\kappa_{\oplus}} B$. We can distinguish two cases.

Case 1: $\kappa_{\oplus}(AD) = \infty$ Because of Lemma 3 we have $\kappa_{\oplus}(AD) = \kappa_{|\Sigma_1}(A) + \kappa'_{|\Sigma_2}(D)$, and with $\kappa'_{|\Sigma_2}(D) = \kappa'(D) < \infty$ we have $\kappa_{|\Sigma_1}(A) = \infty$. Therefore, $\kappa(A) = \kappa_{|\Sigma_1}(A) = \infty$ and thus $A \mathrel{|\!\sim}_{\kappa} B$.

Case 2: $\kappa_{\oplus}(AD) < \infty$ Because $AD \mathrel{|\!\sim}_{\kappa_{\oplus}} B$ we have $\kappa_{\oplus}(ABD) < \kappa_{\oplus}(A\overline{B}D)$. With Lemma 3 we have $\kappa_{\oplus}(ABD) = \kappa_{|\Sigma_1}(AB) + \kappa'_{|\Sigma_2}(D)$ and $\kappa_{\oplus}(A\overline{B}D) = \kappa_{|\Sigma_1}(A\overline{B}) + \kappa'_{|\Sigma_2}(D)$. This implies that $\kappa_{|\Sigma_1}(AB) + \kappa'_{|\Sigma_2}(D) < \kappa_{|\Sigma_1}(A\overline{B}) + \kappa'_{|\Sigma_2}(D)$ and therefore that $\kappa(AB) < \kappa(A\overline{B})$. Thus, $A \mathrel{|\!\sim}_{\kappa} B$. □

Note that for *Direction* $\Leftarrow$ of the proof of Proposition 59 for showing that $A \mathrel{|\!\sim}_{\kappa} B$ we had to pay special attention to the case where $\kappa(D) = \infty$ even though $D \not\mathrel{|\!\sim}^{ec}_{\Delta} \bot$. Therefore, in the proof we employ the extended c-representation $\kappa_{\oplus}$ derived from κ that satisfies $\kappa_{\oplus}(D) < \infty$ and is used to show that $A \mathrel{|\!\sim}_{\kappa} B$.

Combining Propositions 58 and 59 yields that extended c-inference satisfies (SynSplit$^+$):

Proposition 60. *Extended c-inference satisfies (SynSplit$^+$).*

While p-entailment and system Z already do not satisfy syntax splitting for strongly consistent belief bases (Kern-Isberner *et al*., 2020), the results above make c-inference the only other inductive inference operator besides system W (Komo & Beierle, 2020, 2022; Haldimann *et al*., 2023) that has been shown to fully comply with syntax splitting for weakly consistent belief bases.

9. Extended credulous and weakly skeptical c-inference

In the previous sections, we considered skeptical reasoning about the (extended) c-representations of a belief base, that is, the inference operator licenses entailments that hold in all considered ranking functions. In Beierle *et al*. (2016b) two other modes of reasoning are investigated: *credulous* and *weakly skeptical* reasoning. Instantiated for c-representations, these reasoning modes yield the inference operators *credulous c-inference* and *weakly skeptical c-inference*.

Definition 61 (credulous/weakly skeptical c-inference, Beierle *et al.*, 2021). *Let Δ be a strongly consistent belief base and let A, B be formulas.*

- B is a credulous c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{cr}_{\Delta} B$, iff there is a c-representation κ of Δ with $A \mid\!\sim_{\kappa} B$.
- B is a weakly skeptical c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{ws}_{\Delta} B$, iff $A \equiv \bot$ or there is a c-representation κ of Δ with $A \mid\!\sim_{\kappa} B$ and there is no c-representation κ' of Δ with $A \mid\!\sim_{\kappa'} \overline{B}$.

In the following we will explore how these two inference operators can be extended to belief bases that are not strongly consistent by using extended c-representations.

9.1. Extended credulous c-inference

A straightforward way of adapting credulous c-inference to using extended c-representations is to define it as credulous inference over all extended c-representations.

Definition 62 (extended credulous c-inference). *Let Δ be a belief base and let A, B be formulas. B is an extended credulous c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{ecr}_{\Delta} B$, iff (i) Δ is not weakly consistent or (ii) there is an extended c-representation κ of Δ with $A \mid\!\sim_{\kappa} B$.*

Lemma 63. *Extended credulous c-inference is an inductive inference operator, that is, it satisfies (DI) and (TV).*

Proof. For belief bases that are not weakly consistent (DI) holds trivially. For weakly consistent belief bases the proof is analogous to the proof of Proposition 33. □

In comparison to credulous c-inference, extended credulous c-inference is even bolder in the sense that it is based additionally on extended c-representations. This leads to the somewhat unexpected observation that extended credulous c-inference coincides with classical deduction with respect to the material counterparts $A \to B$ of the conditionals $(B|A) \in \Delta$.

Proposition 64. *Let Δ be a belief base and A, B be formulas. Let $\widetilde{\Delta} = \{A_i \to B_i \mid (B_i|A_i) \in \Delta$ be the material counterpart of Δ. Then*

$$A \mid\!\sim^{ecr}_{\Delta} B \quad \textit{iff} \quad \widetilde{\Delta} \models A \to B.$$

Proof. For belief bases Δ that are not weakly consistent, $\widetilde{\Delta}$ is unsatisfiable; then the proposition holds because both $A \mid\!\sim^{ecr}_{\Delta} B$ and $\widetilde{\Delta} \models A \to B$. For weakly consistent belief bases we show both directions of the 'iff'.

Direction $\Rightarrow$ Assume that $A \mid\!\sim^{ecr}_{\Delta} B$. Therefore there is an extended c-representation κ with $A \mid\!\sim_{\kappa} B$, that is, either $\kappa(A) = \infty$ or $\kappa(AB) < \kappa(A\overline{B})$. Let us distinguish these two cases.

Case 1: $\kappa(A) = \infty$

This implies that every model of A has rank ∞. As κ is an extended c-representation, $\kappa(\omega) > 0$ for some ω implies that ω falsifies at least one conditional in Δ. Therefore, all models of A falsify at least one conditional in Δ. Because the models of $\widetilde{\Delta}$ are exactly the worlds that do not falsify any conditional in Δ, we have that $\widetilde{\Delta} \models \neg A$ and therefore $\widetilde{\Delta} \models A \to B$.

Case 2: $\kappa(AB) < \kappa(A\overline{B})$

In this case especially $\kappa(A\overline{B}) > 0$, implying that every model of $A\overline{B}$ falsifies at least one conditional. Therefore, $\widetilde{\Delta} \models \neg(A\overline{B})$ which is equivalent to $\widetilde{\Delta} \models A \to B$.

Direction $\Leftarrow$ Assume that $\widetilde{\Delta} \models A \to B$. Let κ be the extended c-representation obtained from the impact vector $\vec{\eta} = (\infty, \ldots, \infty)$. Then

$$\kappa(\omega) = \begin{cases} 0 & \text{if } \omega \models \widetilde{\Delta} \\ \infty & \text{otherwise.} \end{cases}$$

We will show that $A \mid\!\sim_{\kappa} B$ by distinguishing two cases.

Case 1: $\widetilde{\Delta} \not\models \neg A$

Let $\omega \in Mod_\Sigma(\widetilde{\Delta} \cup \{A\})$. We have that $\kappa(\omega) = 0$. Because $\widetilde{\Delta} \models A \to B$, we know that $\omega \models B$. Hence, $\kappa(AB) = 0$. Furthermore, $\widetilde{\Delta} \models A \to B$ yields that $\kappa(A\overline{B}) > 0$. Thus, $\kappa(AB) < \kappa(A\overline{B})$ and $A \mid\!\sim_\kappa B$.

Case 2: $\widetilde{\Delta} \models \neg A$

In this case, $\kappa(A) = \infty$ and thus $A \mid\!\sim_\kappa B$. □

It should also be noted that for strongly consistent belief bases, every credulous c-inference is also an extended credulous c-inference, but not the other way round.

Proposition 65. *(1) For every strongly consistent* Δ *and formulas A, B, if* $A \mid\!\sim^{cr}_\Delta B$ *then* $A \mid\!\sim^{ecr}_\Delta B$*. (2) There are* Δ, A, B *such that* $A \not\mid\!\sim^{cr}_\Delta B$ *but* $A \mid\!\sim^{ecr}_\Delta B$*.*

Proof. *Ad (1):* This follows directly from Proposition 22 and the definitions of credulous c-inference and extended credulous c-inference.

Ad (2): Consider $\Delta = \{(b|p), (f|b), (\overline{f}|p)\}$ over $\Sigma = \{p, b, f\}$. Then $p \not\mid\!\sim^{cr}_\Delta f$ but $p \mid\!\sim^{ecr}_\Delta f$ because for $\vec{\eta} = (\infty, \infty, \infty)$ we have $\kappa_{\vec{\eta}}(p) = \infty$ and thus $p \mid\!\sim_{\kappa_{\vec{\eta}}} f$. □

Furthermore, extended credulous c-inference fails to satisfy (Classic Preservation). For instance, continuing the example in the proof of Proposition 65 (2), we have $p \not\mid\!\sim^{p}_\Delta \perp$ but $p \mid\!\sim^{ecr}_\Delta \perp$ because $\kappa_{\vec{\eta}}(p) = \infty$ for $\vec{\eta} = (\infty, \infty, \infty)$. This motivates the introduction of a version of extended credulous c-inference which coincides with credulous c-inference on strongly consistent belief bases and which complies with (Classic Preservation). This can be achieved by taking only the conservative extended c-representations in $CMod^{ec}_\Delta$ into account.

Definition 66 (conservative extended credulous c-inference). *Let* Δ *be a belief base and let A, B be formulas. B is a conservative extended credulous c-inference from A in the context of* Δ*, denoted by* $A \mid\!\sim^{cecr}_\Delta B$*, iff (i)* Δ *is not weakly consistent or (ii) there is a conservative extended c-representation* $\kappa \in CMod^{ec}_\Delta$ *of* Δ *with* $A \mid\!\sim_\kappa B$*.*

Lemma 67. *Conservative extended credulous c-inference is an inductive inference operator, that is, it satisfies (DI) and (TV).*

Proof. Analogous to the proof of Proposition 33. □

Lemma 68. *For strongly consistent belief bases, conservative extended credulous c-inference coincides with credulous c-inference.*

Proof. Let Δ be a strongly consistent belief base. In this case we have $\kappa^z_\Delta(\omega) < \infty$ for all $\omega \in \Omega$. By definition of $CMod^{ec}_\Delta$ (cf. Definition 43), we have $\kappa(\omega) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$ and for all $\omega \in \Omega$. Thus $CMod^{ec}_\Delta$ coincides with the set of c-representations of Δ, and the inferences that can be made with extended credulous c-inference from Δ coincide with the inferences that can be made with credulous c-inference from Δ. □

Proposition 69. *Conservative extended credulous c-inference satisfies (Classic Preservation).*

Proof. We have to show that $A \mid\!\sim^{cecr}_\Delta \perp$ iff $A \mid\!\sim^{p}_\Delta \perp$. Using Lemma 14, it suffices to show that $A \mid\!\sim^{cecr}_\Delta \perp$ iff $\kappa^z_\Delta(A) = \infty$. Combining Definition 43 and Lemma 24 yields that $\kappa(\omega) = \infty$ iff $\kappa^z_\Delta(\omega) = \infty$ for every $\kappa \in CMod^{ec}_\Delta$ and every $\omega \in \Omega$. Thus, the proposition holds. □

In the following we will show how conservative extended credulous c-inference can be realized by a CSP. This approach is inspired by the CSP characterization of credulous c-inference in Beierle *et al.* (2019b) and the CSP realization of extended (skeptical) c-inference in Subsection 7.2. We will again use the CSP $CRS^{ex}_\Sigma(\Delta)$ characterizing all extended c-inferences in $CMod^{ec}_\Delta$ (cf. Proposition 47).

The following constraint encodes that $A \mid\!\sim_{\kappa_{\vec{\eta}}} B$ holds for an extended c-representation $\kappa_{\vec{\eta}}$.

Definition 70. *Let* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ *be a belief base and let* J_Δ *be as defined in Definition 45. The constraint* $CR_\Delta(B|A)$ *is given by*

$$\min_{\omega \models AB} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i \;<\; \min_{\omega \models A\overline{B}} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i. \tag{11}$$

By combining this constraint with the CSP $CRS^{ex}_\Sigma(\Delta)$, we can check whether $A \mid\!\sim^{cecr}_\Delta B$ with the following proposition.

Proposition 71. *Let Δ be a weakly consistent belief base. Then $A \mid\!\sim^{cecr}_\Delta B$ iff either (i) $\kappa^z_\Delta(A\overline{B}) = \infty$, or (ii) $\kappa^z_\Delta(A\overline{B}) < \infty$, $\kappa^z_\Delta(AB) < \infty$ and $CRS^{ex}_\Sigma(\Delta) \cup CR_\Delta(B|A)$ is solvable.*

Proof. **Direction** $\Rightarrow$ Assume that $A \mid\!\sim^{cecr}_\Delta B$ and that $\kappa^z_\Delta(A\overline{B}) < \infty$. Then $\kappa(A\overline{B}) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$ by the definition of $CMod^{ec}_\Delta$. Therefore, $\kappa(A) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$. Because $A \mid\!\sim^{cecr}_\Delta B$, there is at least one ranking function $\kappa^* \in CMod^{ec}_\Delta$ with $A \mid\!\sim_{\kappa^*} B$ and thus $\kappa^*(AB) < \kappa^*(A\overline{B})$. This implies that $\kappa^*(AB) < \infty$ and, by the definition of $CMod^{ec}_\Delta$, $\kappa^z_\Delta(AB) < \infty$. Because of Proposition 47 there must be an $\vec{\eta}^* \in Sol^{J+\infty}_\Delta$ with $\kappa^* = \kappa_{\vec{\eta}^*}$ and thus $\kappa_{\vec{\eta}^*}(AB) < \kappa_{\vec{\eta}^*}(A\overline{B})$. We have

$$\begin{aligned}
& \kappa_{\vec{\eta}^*}(AB) < \kappa_{\vec{\eta}^*}(A\overline{B}) \\
\Leftrightarrow\quad & \min_{\omega \models AB} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta^*_i < \min_{\omega \models A\overline{B}} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta^*_i \\
\overset{(*)}{\Leftrightarrow}\quad & \min_{\omega \models AB} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta^*_i \;<\; \min_{\omega \models A\overline{B}} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta^*_i.
\end{aligned}$$

Equivalence $(*)$ holds because the ranks of the minimal models of $A\,B$ and $A\overline{B}$ are finite and therefore do not violate a conditional $(B_i|A_i)$ with $i \notin J_\Delta$.

Therefore, $CR_\Delta(B|A)$ holds for the solution of $CRS^{ex}_\Sigma(\Delta)$ corresponding to $\vec{\eta}^*$, implying that $CRS^{ex}_\Sigma(\Delta) \cup CR_\Delta(B|A)$ is solvable.

Direction $\Leftarrow$ Assume that either $\kappa^z_\Delta(A\overline{B}) = \infty$ or $\big(\kappa^z_\Delta(A\overline{B}) < \infty$, $\kappa^z_\Delta(AB) < \infty$ and $CRS^{ex}_\Sigma(\Delta) \cup CR_\Delta(B|A)$ is solvable$\big)$. There are three cases.

Case 1: $\kappa^z_\Delta(AB) = \infty$ and $\kappa^z_\Delta(A\overline{B}) = \infty$

Then $\kappa^z_\Delta(A) = \infty$ and, by Proposition 26, $\kappa(A) = \infty$ for every $\kappa \in CMod^{ec}_\Delta$. Because $CMod^{ec}_\Delta$ contains at least one c-representation, $A \mid\!\sim^{ecr}_\Delta B$.

Case 2: $\kappa^z_\Delta(AB) < \infty$ and $\kappa^z_\Delta(A\overline{B}) = \infty$

Then, by the definition of $CMod^{ec}_\Delta$, we have $\kappa(AB) < \infty$ and, by Proposition 26, $\kappa(A\overline{B}) = \infty$ for every $\kappa \in CMod^{ec}_\Delta$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_\Delta$, and because $CMod^{ec}_\Delta$ contains at least one c-representation, $A \mid\!\sim^{cecr}_\Delta B$.

Case 3: $\kappa^z_\Delta(A\overline{B}) < \infty$

Then, by assumption, $CRS^{ex}_\Sigma(\Delta) \cup CR_\Delta(B|A)$ is solvable and $\kappa^z_\Delta(AB) < \infty$. This implies that there is an $\vec{\eta}^J \in Sol(CRS^{ex}_\Sigma(\Delta))$ satisfying $CR_\Delta(B|A)$. Using the equivalence transformations in the part of the proof for *Direction* $\Rightarrow$, there is an $\vec{\eta}^* \in Sol^{J+\infty}_\Delta$ constructed from $\vec{\eta}^J$ with $\kappa_{\vec{\eta}^*}(AB) < \kappa_{\vec{\eta}^*}(A\overline{B})$. With Proposition 47, it follows that $A \mid\!\sim^{cecr}_\Delta B$. □

9.2. Extended weakly skeptical c-inference

Analogous to extended credulous c-inference, we can define extended weakly skeptical c-inference as weakly skeptical inference over all extended c-representations.

Definition 72 (extended weakly skeptical c-inference). *Let Δ be a belief base and let A, B be formulas. B is an extended weakly skeptical c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{ews}_\Delta B$, iff (i) Δ is not weakly consistent, (ii) there is an extended c-representation κ° of Δ with $\kappa^\circ(A) = \infty$, or (iii) there is an extended c-representation κ of Δ with $A \mid\!\sim_\kappa B$ and there is no extended c-representation κ' of Δ with $A \mid\!\sim_{\kappa'} \overline{B}$.*

Lemma 73. *Extended weakly skeptical c-inference is an inductive inference operator, that is, it satisfies (DI) and (TV).*

Proof. **(DI):** Let $(B_i|A_i) \in \Delta$. We need to show that $A_i \mid\!\sim^{ews}_{\Delta} B_i$. If Δ is not weakly consistent or if there is an extended c-representation κ° with $\kappa^\circ(A_i) = \infty$, then $A_i \mid\!\sim^{ews}_{\Delta} B_i$ by definition. For the remainder of the proof assume that Δ is weakly consistent and $\kappa(A_i) < \infty$ for every extended c-representation κ of Δ. Every extended c-representation κ is a model of $(B_i|A_i)$ and therefore must satisfy $\kappa(A_iB_i) < \kappa(A_i\overline{B_i})$. Thus, $A_i \mid\!\sim^{ews}_{\Delta} B_i$.

(TV): Analogous to the proof of Proposition 33. □

Interestingly, this inference operator also coincides with classical deduction with respect to the material counterparts of the conditionals.

Proposition 74. *Let Δ be a belief base and A, B be formulas. Let $\widetilde{\Delta} = \{A_i \to B_i \mid (B_i|A_i) \in \Delta$.*

$$A \mid\!\sim^{ews}_{\Delta} B \quad \text{iff} \quad \widetilde{\Delta} \models A \to B.$$

Proof. For belief bases Δ that are not weakly consistent, $\widetilde{\Delta}$ is unsatisfiable; then the proposition holds because both $A \mid\!\sim^{ews}_{\Delta} B$ and $\widetilde{\Delta} \models A \to B$. For weakly consistent belief bases we show both directions of the 'iff'.

Direction $\Rightarrow$ Assume that $A \mid\!\sim^{ews}_{\Delta} B$. We can distinguish two cases.

Case 1: There is an extended c-representation κ° with $\kappa^\circ(A) = \infty$

Then every model of A falsifies at least one conditional in Δ. But by the construction of $\widetilde{\Delta}$, the models of $\widetilde{\Delta}$ do not falsify any conditional in Δ. Thus, $\widetilde{\Delta} \models \neg A$ and therefore $\widetilde{\Delta} \models A \to B$.

Case 2: There is no such κ° with $\kappa^\circ(A) = \infty$

Then there must be an extended c-representation κ with $A \mid\!\sim_{\kappa} B$. In this case it cannot be that $\kappa(A) = \infty$, therefore we have $\kappa(AB) < \kappa(A\overline{B})$. Especially, we have $\kappa(A\overline{B}) > 0$, implying that every model of $A\overline{B}$ falsifies at least one conditional and thus is not a model of $\widetilde{\Delta}$. Therefore, $\widetilde{\Delta} \models \neg(A\overline{B})$ which is equivalent to $\widetilde{\Delta} \models A \to B$.

Direction $\Leftarrow$ Assume that $\widetilde{\Delta} \models A \to B$. Let κ be the extended c-inference obtained from the impact vector $\vec{\eta} = (\infty, \ldots, \infty)$. Then

$$\kappa(\omega) = \begin{cases} 0 & \text{if } \omega \models \widetilde{\Delta} \\ \infty & \text{otherwise.} \end{cases}$$

We can distinguish two cases.

Case 1: $\widetilde{\Delta} \cup \{A\}$ is satisfiable

Then there is at least one model ω of A that does not falsify any conditional. Because $\widetilde{\Delta} \models A \to B$ we have $\omega \models AB$. Therefore, $\kappa(AB) = 0$. Furthermore, every model of $A\overline{B}$ must falsify at least one conditional, implying that $\kappa(A\overline{B}) = \infty$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ and $A \mid\!\sim_{\kappa} B$.

Additionally, because $\widetilde{\Delta} \models A \to B$ we have $\kappa'(AB) = 0$ for every extended c-representation κ' of Δ. This implies that there is no extended c-representation κ' with $A \mid\!\sim_{\kappa} \overline{B}$. In summary, $A \mid\!\sim^{ews}_{\Delta} B$.

Case 2: $\widetilde{\Delta} \cup \{A\}$ is not satisfiable

Then every model of A falsifies at least one conditional in Δ, implying that $\kappa(A) = \infty$. This implies $A \mid\!\sim^{ews}_{\Delta} B$. □

Thus, credulous and weakly skeptical c-inference over all extended c-representations coincide because

$$A \mid\!\sim^{ecr}_{\Delta} B \quad \text{iff} \quad A \mid\!\sim^{ews}_{\Delta} B$$

according to Propositions 64 and 74. Moreover, analogously to extended credulous c-inference (Proposition 65), we observe that for strongly consistent belief bases, extended weakly skeptical c-inference extends weakly skeptical c-inference but does not coincide with it.

Proposition 75. *(1) For every strongly consistent Δ and formulas A, B, if $A \mid\!\sim^{ws}_{\Delta} B$ then $A \mid\!\sim^{ews}_{\Delta} B$. (2) There are Δ, A, B such that $A \not\mid\!\sim^{ws}_{\Delta} B$ but $A \mid\!\sim^{ews}_{\Delta} B$.*

Proof. *Ad (1):* Let $A \mid\!\sim^{ws}_{\Delta} B$. This implies that $A \mid\!\sim^{cr}_{\Delta} B$ (Beierle *et al.*, 2021) and with Proposition 65 that $A \mid\!\sim^{ecr}_{\Delta} B$. By Propositions 64 and 74 we have $A \mid\!\sim^{ews}_{\Delta} B$.

Ad (2): Consider $\Delta = \{(b|p), (f|b), (\overline{f}|p)\}$ over $\Sigma = \{p, b, f\}$. Then $p \not\mid\!\sim^{ws}_{\Delta} f$ but $p \mid\!\sim^{ews}_{\Delta} f$ because for $\vec{\eta} = (\infty, \infty, \infty)$ we have $\kappa_{\vec{\eta}}(p) = \infty$. □

Furthermore, similar to the credulous case, also extended weakly skeptical inference fails to satisfy (Classic Preservation). For instance, continuing the example in the proof of Proposition 75 (2), we have $p \not\mid\!\sim^{p}_{\Delta} \bot$ but $p \mid\!\sim^{ecr}_{\Delta} \bot$ because $\kappa_{\vec{\eta}}(p) = \infty$ for $\vec{\eta} = (\infty, \infty, \infty)$. Therefore, we consider weakly skeptical inference only over the conservative extended c-representations of a belief base.

Definition 76 (conservative extended weakly skeptical c-inference). *Let Δ be a belief base and let A, B be formulas. B is a conservative extended weakly skeptical c-inference from A in the context of Δ, denoted by $A \mid\!\sim^{cews}_{\Delta} B$, iff (i) Δ is not weakly consistent, (ii) there is a conservative extended c-representation κ° of Δ with $\kappa^\circ(A) = \infty$, or (iii) there is a conservative extended c-representation κ of Δ with $A \mid\!\sim_{\kappa} B$ and there is no conservative extended c-representation κ' of Δ with $A \mid\!\sim_{\kappa'} \overline{B}$.*

Lemma 77. *Conservative extended weakly skeptical c-inference is an inductive inference operator, that is, it satisfies (DI) and (TV).*

Proof. **(DI):** Let $A \mid\!\sim^{p}_{\Delta} B$. If Δ is not weakly consistent or if there is a conservative extended c-representation κ° with $\kappa^\circ(A) = \infty$ then $A \mid\!\sim^{cews}_{\Delta} B$ follows immediately. Assume for the remainder of this proof that Δ is weakly consistent and $\kappa(A) < \infty$ for all conservative extended c-representations κ of Δ. For every conservative extended c-representations κ of Δ it holds that $A \mid\!\sim_{\kappa} B$. Because $\kappa(A) < \infty$ this implies that $\kappa(AB) < \kappa(A\overline{B})$. Thus, $A \not\mid\!\sim_{\kappa} \overline{B}$. In summary we have $A \mid\!\sim^{cews}_{\Delta} B$.

(TV): Analogous to the proof of Proposition 33. □

Lemma 78. *For strongly consistent belief bases, conservative extended weakly skeptical c-inference coincides with weakly skeptical c-inference.*

Proof. Let Δ be a strongly consistent belief base. In this case we have $\kappa^{z}_{\Delta}(\omega) < \infty$ for all $\omega \in \Omega$. By definition of $CMod^{ec}_{\Delta}$ (cf. Definition 43), it is $\kappa(\omega) < \infty$ for all $\kappa \in CMod^{ec}_{\Delta}$ and $\omega \in \Omega$. We have $\kappa(A) = \infty$ iff $A \equiv \bot$ for all $\kappa \in CMod^{ec}_{\Delta}$. Thus, $CMod^{ec}_{\Delta}$ coincides with the set of c-representations of Δ, and the inferences that can be made with conservative extended weakly skeptical c-inference from Δ coincide with the inferences that can be made with weakly skeptical c-inferences from Δ. □

Proposition 79. *Conservative extended weakly skeptical c-inference satisfies (Classic Preservation).*

Proof. We have to show that $A \mid\!\sim^{cews}_{\Delta} \bot$ iff $A \mid\!\sim^{p}_{\Delta} \bot$. For Δ not weakly consistent, we have both $A \mid\!\sim^{cews}_{\Delta} \bot$ and $A \mid\!\sim^{p}_{\Delta} \bot$ and thus the proposition holds. For weakly consistent Δ, using Lemma 14, it suffices to show that $A \mid\!\sim^{cews}_{\Delta} \bot$ iff $\kappa^{z}_{\Delta}(A) = \infty$. Combining Definition 43 and Lemma 24 yields that $\kappa(\omega) = \infty$ iff $\kappa^{z}_{\Delta}(\omega) = \infty$ for every $\kappa \in CMod^{ec}_{\Delta}$ and every $\omega \in \Omega$. Thus, the proposition holds. □

We will now realize conservative extended weakly skeptical c-inference by a CSP. Inspired by the CSP characterization of weakly skeptical c-inference in Beierle *et al*. (2019b) we will again use the CSPs $CRS^{ex}_{\Sigma}(\Delta)$ and $CR_{\Delta}(B|A)$ and $\neg CR_{\Delta}(B|A)$ as in Propositions 71 and 51.

Proposition 80. *Let Δ be a weakly consistent belief base. Then $A \mid\!\sim^{cews}_{\Delta} B$ iff either (i) $\kappa^{z}_{\Delta}(A\overline{B}) = \infty$ or (ii) $\kappa^{z}_{\Delta}(A\overline{B}) < \infty$, $\kappa^{z}_{\Delta}(AB) < \infty$ and $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(B|A)$ is solvable and $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(\overline{B}|A)$ is unsolvable.*

Proof. **Direction** $\Rightarrow$ Assume that $A \mid\!\sim^{cews}_{\Delta} B$ and that $\kappa^{z}_{\Delta}(A\overline{B}) < \infty$. Then $\kappa(A\overline{B}) < \infty$ for all $\kappa \in CMod^{ec}_{\Delta}$ by the definition of $CMod^{ec}_{\Delta}$. Therefore, $\kappa(A) < \infty$ for all $\kappa \in CMod^{ec}_{\Delta}$.

Because $A \mid\!\sim^{cews}_{\Delta} B$, there is at least one ranking function $\kappa^* \in CMod^{ec}_{\Delta}$ with $A \mid\!\sim_{\kappa^*} B$ and thus $\kappa^*(AB) < \kappa^*(A\overline{B})$. This implies that $\kappa^*(AB) < \infty$ and, by Proposition 26, $\kappa^{z}_{\Delta}(AB) < \infty$. Because of Proposition 74 there must be an $\vec{\eta}^* \in Sol^{J+\infty}_{\Delta}$ with $\kappa^* = \kappa_{\vec{\eta}^*}$ and thus $\kappa_{\vec{\eta}^*}(AB) < \kappa_{\vec{\eta}^*}(A\overline{B})$. Analogously to the proof of Proposition 71, we can show that this is equivalent to the satisfication of $CR_{\Delta}(B|A)$. Therefore, $CR_{\Delta}(B|A)$ holds for the solution of $CRS^{ex}_{\Sigma}(\Delta)$ corresponding to $\vec{\eta}^*$, implying that $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(B|A)$ is solvable.

$A \mathrel{|\!\!\sim}^{cews}_{\Delta} B$ also implies that there is no conservative extended c-representation $\kappa' \in CMod^{ec}_{\Delta}$ with $A \mathrel{|\!\!\sim}_{\kappa'} \overline{B}$. This implies that there is no κ' with $\kappa'(A\overline{B}) < \kappa'(AB)$. Because of Proposition 47 there cannot be an $\vec{\eta}' \in Sol^{J+\infty}_{\Delta}$ with $\kappa_{\vec{\eta}'}(A\overline{B}) < \kappa_{\vec{\eta}'}(AB)$. Analogously to the proof of Proposition 71, we can show that $\kappa_{\vec{\eta}}(A\overline{B}) < \kappa_{\vec{\eta}}(AB)$ is equivalent to $CR_{\Delta}(\overline{B}|A)$. Therefore, there cannot be a solution of $CRS^{ex}_{\Sigma}(\Delta)$ that satisfies $CR_{\Delta}(\overline{B}|A)$, implying that $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(\overline{B}|A)$ is unsolvable.

Direction $\Leftarrow$ Assume that either $\kappa^{z}_{\Delta}(A\overline{B}) = \infty$, or $\big(\kappa^{z}_{\Delta}(A\overline{B}) < \infty$, $\kappa^{z}_{\Delta}(AB) < \infty$ and $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(B|A)$ is solvable and $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(\overline{B}|A)$ is unsolvable$\big)$. There are three cases.

Case 1: $\kappa^{z}_{\Delta}(AB) = \infty$ and $\kappa^{z}_{\Delta}(A\overline{B}) = \infty$

Then $\kappa^{z}_{\Delta}(A) = \infty$. Because Δ is weakly consistent there is at least one conservative extended c-representation $\kappa \in CMod^{ec}_{\Delta}$ and, by Proposition 26, $\kappa(A) = \infty$. Therefore, $A \mathrel{|\!\!\sim}^{cews}_{\Delta} B$.

Case 2: $\kappa^{z}_{\Delta}(AB) < \infty$ and $\kappa^{z}_{\Delta}(A\overline{B}) = \infty$

Then, by the definition of $CMod^{ec}_{\Delta}$, we have $\kappa(AB) < \infty$ and, by Proposition 26, $\kappa(A\overline{B}) = \infty$ for every $\kappa \in CMod^{ec}_{\Delta}$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_{\Delta}$. Because Δ is weakly consistent there is at least one $\kappa^* \in CMod^{ec}_{\Delta}$ with $\kappa^*(AB) < \kappa^*(A\overline{B})$ and thus $A \mathrel{|\!\!\sim}_{\kappa^*} B$. Furthermore, there cannot be a conservative extended c-representation κ' with $A \mathrel{|\!\!\sim}_{\kappa'} \overline{B}$, because this would imply $\kappa'(A) = \infty$ or $\kappa'(A\overline{B}) < \kappa'(AB)$, contradicting the results above. Thus, $A \mathrel{|\!\!\sim}^{cews}_{\Delta} B$.

Case 3: $\kappa^{z}_{\Delta}(A\overline{B}) < \infty$

Then, by assumption, $\kappa^{z}_{\Delta}(AB) < \infty$, the CSP $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(B|A)$ is solvable, and $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(\overline{B}|A)$ is unsolvable. $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(B|A)$ being solvable implies that there is an $\vec{\eta}^J \in Sol(CRS^{ex}_{\Sigma}(\Delta))$ satisfying $CR_{\Delta}(B|A)$. Using the equivalence transformations as in the proof for Proposition 71, *Direction* $\Rightarrow$, there is an $\vec{\eta}^* \in Sol^{J+\infty}_{\Delta}$ constructed from $\vec{\eta}^J$ with $\kappa_{\vec{\eta}^*}(AB) < \kappa_{\vec{\eta}^*}(A\overline{B})$. With Proposition 47, it follows that $\kappa_{\vec{\eta}^*} \in CMod^{ec}_{\Delta}$. Thus, we found a conservative extended c-representation $\kappa^* = \kappa_{\vec{\eta}^*}$ with $A \mathrel{|\!\!\sim}_{\kappa^*} B$.

Analogously, $CRS^{ex}_{\Sigma}(\Delta) \cup CR_{\Delta}(\overline{B}|A)$ being unsolvable implies that $CR_{\Delta}(\overline{B}|A)$ is violated for every solution of $CRS^{ex}_{\Sigma}(\Delta)$. Using the equivalence transformation from the proof for Proposition 71, *Direction* $\Rightarrow$, this implies that for every $\vec{\eta} \in Sol^{J+\infty}_{\Delta}$ we have $\kappa_{\vec{\eta}}(AB) \not< \kappa_{\vec{\eta}}(A\overline{B})$ and thus, with Proposition 47, $\kappa(AB) \not< \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_{\Delta}$. Therefore, there cannot be a conservative extended c-representation κ' with $A \mathrel{|\!\!\sim}_{\kappa'} \overline{B}$, because this would imply $\kappa'(A) = \infty$ or $\kappa'(A\overline{B}) < \kappa'(AB)$, contradicting the results above.

In summary, it follows that $A \mathrel{|\!\!\sim}^{cews}_{\Delta} B$. □

The inductive inference operators developed and addressed in this article together with their interrelationships are summarized and illustrated in Figure 2.

10. Conclusions and future work

Weakly consistent belief bases are belief bases that may enforce some worlds to be completely implausible. In this article, we investigated how the knowledge given in a conditional belief base that is not strongly consistent, but only weakly consistent, can be completed by applying an inductive inference operator. After presenting the treatment of weakly consistent belief bases by p-entailment and system Z, we focussed on c-representations. For c-inference extended to weakly consistent belief bases, we developed a CSP characterizing it. We showed that extended c-inference can be realized equivalently by a specialized CSP taking just conservative extended c-representations into account. Moreover, we showed that extended c-inference inherits highly desirable inference properties, in particular, that extended c-inference fully complies with syntax splitting. For this, we proved that extended c-inference satisfies adapted versions of (Rel) and (Ind) that also take inference from weakly consistent belief bases into account. We also extended credulous and weakly skeptical c-inference and provided realizations for them as CSPs. We observed that for credulous and weakly skeptical c-inference, the distinction whether extended or only conservative extended c-representations are taken into account is significant, leading to a map of interesting, specific interrelationsships among the various inference operators. In Haldimann

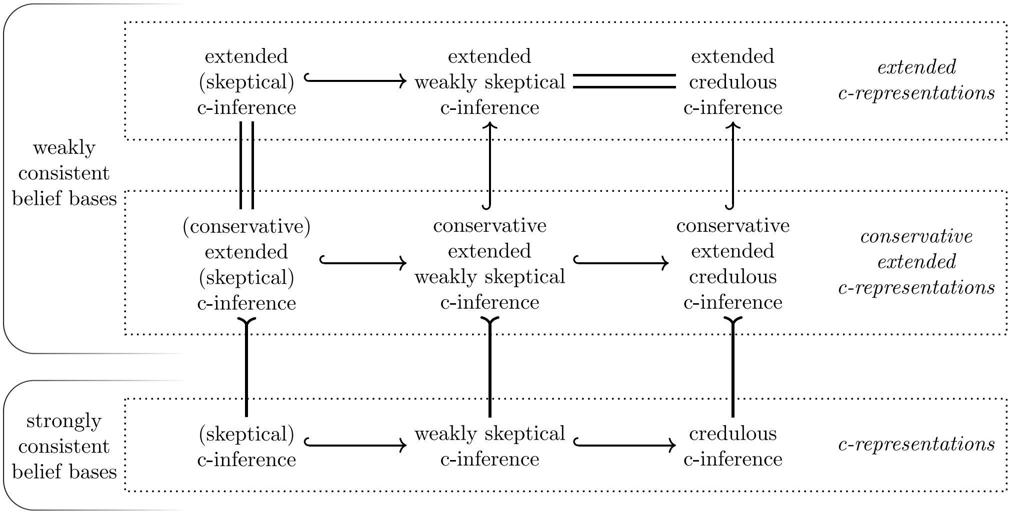

Figure 2. *A Summary of inductive inference operators and their interrelationships. The left-most column indicates whether the inference operator is defined on weakly consistent belief bases or on strongly consistent belief bases only. The right-most column indicates the class of c-representations that the inference operator takes into account. An arrow $I_1 \multimap I_2$ indicates that the inductive inference operator I_1 and the restriction of I_2 to strongly consistent belief bases coincide. A double line $I_1 = I_2$ indicates that I_1 and I_2 coincide. An arrow $I_1 \hookrightarrow I_2$ indicates that I_1 is captured by I_2 and that I_1 is strictly extended by I_2 for some belief bases.*

and Beierle (2024), a comprehensive map of the relationships of extended c-inference to other inductive inference operators taking weakly consistent belief bases into account can be found.

Future work includes investigating how the reasoning with infeasible worlds relates to approaches to paraconsistent reasoning like Priest's logic LP (Priest, 1979). Furthermore, similarly as it has been done for c-inference, we will employ compilation techniques (Beierle *et al.*, 2019b) for extended c-inference and we will realize it as a SAT and as an SMT problem (Beierle *et al.*, 2022; von Berg *et al.*, 2023, 2024) and implement it in the InfOCF reasoning platform (Beierle *et al.*, 2017; Kutsch & Beierle, 2021; Beierle *et al.*, 2024, 2025). We will also generalize the implementations of p-entailment and system Z existing in InfOCF to the treatment of weakly consistent belief bases as described in this article.

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